Math 130 04 – A Survey of Calculus

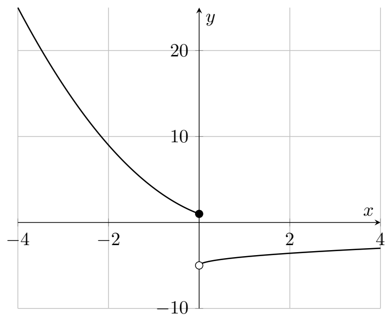
Take-home test 1

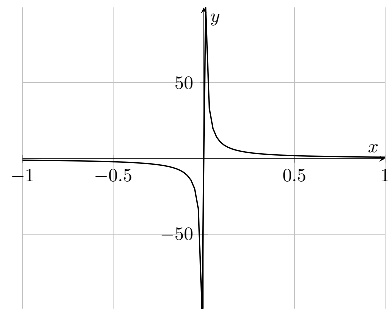
Due: Tuesday, October 11, 2022, 11:59PM (**hard deadline**)

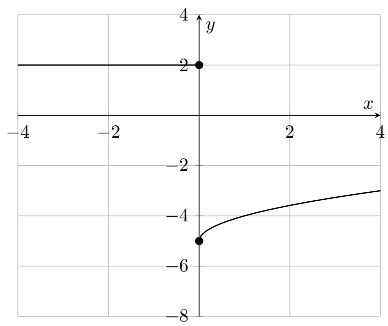
Instructions:

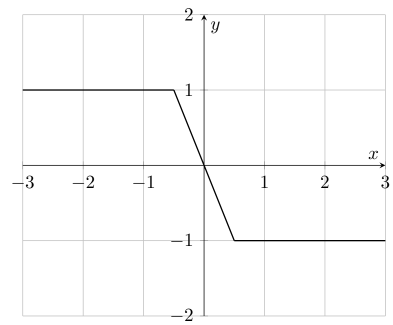
- This test has **four** questions, each worth **five** points. Your goal is to get **16** points in total.
- Any extra points (> 16) will eventually count towards increasing your grade (A→A^+, B^+→A, B^-→B, and so on) at the end of the semester.
- Each question is divided into subquestions. The points that each subquestion is worth are indicated next to it.
- Write your answers on separate sheets of paper.
- Write your name at the top of **each** page you use, and number each page.
- Number your answers correctly.
- **Justify each step in all your answers fully and clearly.** Answers with no explanation (*even if the calculation is correct*) are worth **zero** points. Answers with a full and correct explanation but a calculation error are worth more than 90% of the points.
- You are expected to work on this test **alone**. Plagiarism will be sanctioned with a fail grade.

1. (a) 2 points Which of the following graphs represent real functions? Which of the functions is continuous over the interval $[-1, 1]$?

i. 

iii. 

ii. 

iv. 

(b) 3 points Calculate the following limits. (Hint: you can use a graphing calculator to see what the functions look like, but you should also be able to justify your calculation.)

i. $\lim_{x\to 3} x^3 + 4x^2 + 2$ ii. $\lim_{x\to 0} \frac{x^2+3x}{4x^3+3x^2+5x}$ iii. $\lim_{x\to\infty} \frac{13x^2}{x^3+2x^2+5x}$

2. Consider the following definition.

$$f(x) = \begin{cases} \dfrac{2}{x^5+2x^2-1} & \text{if } x \le 0 \\ x^3+4x-2 & \text{if } x > 0 \end{cases}$$

(a) 1.5 points Is f a real function? If so, what is its domain?

(b) 1.5 points Do the following limits exist? Justify your answer. If they do exist, calculate them.

i. $\lim_{x\to -1} f(x)$ ii. $\lim_{x\to 0^-} f(x)$ iii. $\lim_{x\to 0} f(x)$

(c) 2 points Consider the definition $g(x) = (7x^3 + 10x^2 + 5x + 2) \cdot f(x)$.

i. When $x \le 0$, is $g(x)$ equal to a rational function?

ii. Calculate $\lim_{x\to -1} g(x)$

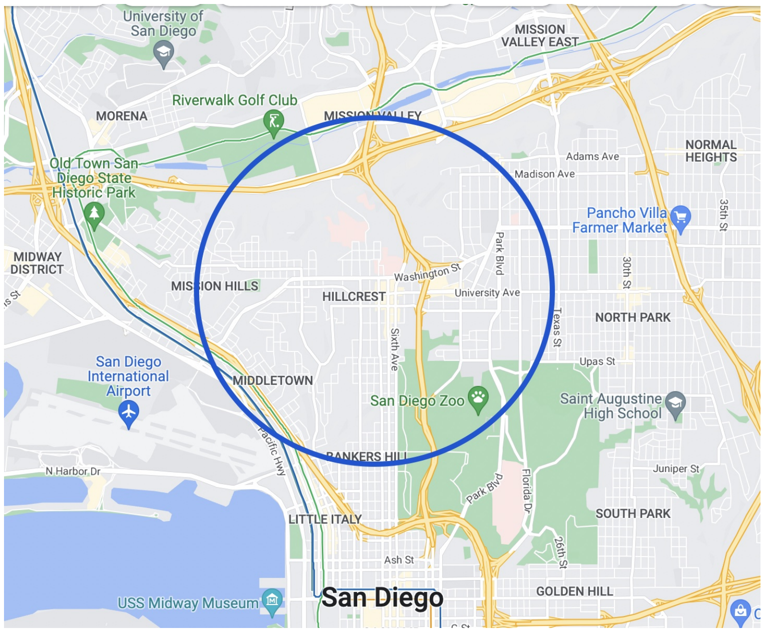
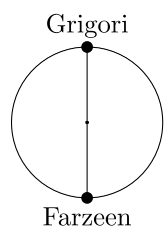


Figure 1: A random circle on a map.

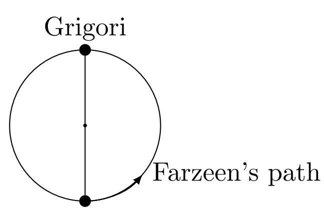
3. A (seemingly crazy) mathematician bets you \$1000 that if you draw a random circle (of any size) on any map, there are always two points on the circle that are at exactly the same elevation (height above sea level) and that are diametrically opposite each other (i.e. such that the straight line joining the two points passes through the center). Your goal is to figure out if you should take the bet.

   Let Farzeen and Grigori be two people standing at any two diametrically opposite points of the circle.

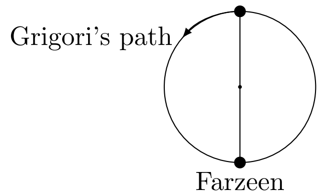


   So, in the example of a random circle on a map in Figure 1, Farzeen would be on Bankers Hill and Grigori would be in Mission Valley.

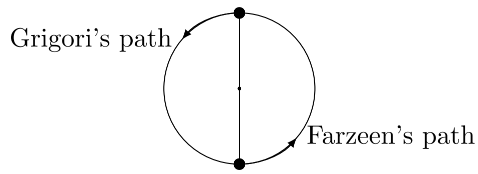
   (a) [1 point] Suppose Farzeen walks along the circle towards Grigori, and reaches him after 30 minutes. Let $f(x)$ be Farzeen's elevation (in feet above sea level) at x minutes. Is f a continuous function on the interval $[0, 30]$?



   (b) [1 point] Suppose instead that Grigori walks along the *other* side of the circle towards Farzeen, and reaches her after 30 minutes. Let $g(x)$ be Grigori's elevation (in feet above sea level) at x minutes. Is g a continuous function on the interval $[0, 30]$?



(c) 1.5 points Now suppose Farzeen and Grigori walk towards each other's starting point, *always* staying on diametrically opposite points of the circle at any time x, and suppose each of them ends up at the other's starting point after 30 minutes. As before, let $f(x)$ be Farzeen's elevation after x minutes and let $g(x)$ be Grigori's elevation after x minutes.



i. If $f(0) > g(0)$, what can you say about about $f(30) - g(30)$?

ii. If $f(0) < g(0)$, what can you say about about $f(30) - g(30)$?

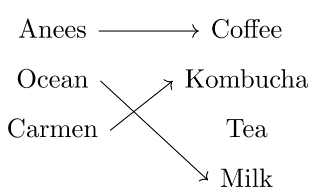
(d) 1.5 points Is there some time x between 0 and 30 minutes such that $f(x) = g(x)$? (Farzeen and Grigori are at exactly the same elevation but on opposite sides of the circle.) Justify your answer. Should you take the crazy mathematician's bet?

4. A function $f\colon A \to B$ from a set A to a set B is **injective** if no two distinct elements of A are sent to the same element of B.

In other words, a function $f\colon A \to B$ is an injective function if:

$$\text{whenever } a_1 \in A \text{ and } a_2 \in A \text{ and } a_1 \neq a_2, \\ \text{then } f(a_1) \neq f(a_2).$$

(a) 1 point Which of the following correspondences are injective functions?

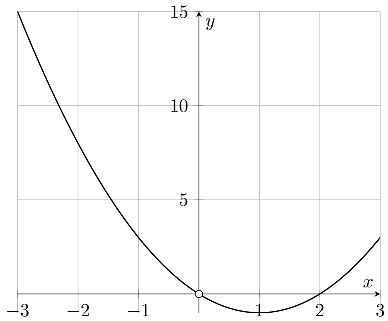
i. 

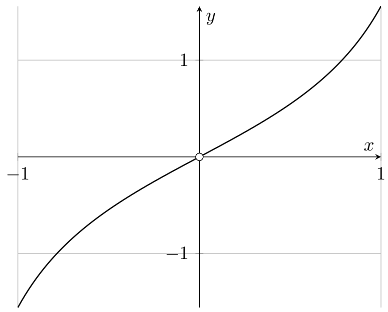
ii. $A =$ the set of all spectators at a soccer game,
$B =$ the set of all seats in the stadium,
$f\colon A \nrightarrow B$ is the correspondence that maps every spectator to their seat.

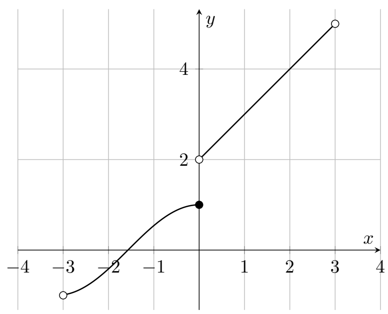
A function is injective exactly when its graph satisfies the following test.

Horizontal Line Test: Every horizontal line passes through *at most* one point of the graph.

(b) 1.5 points Which of the following graphs represent injective functions?

i. 

ii. 

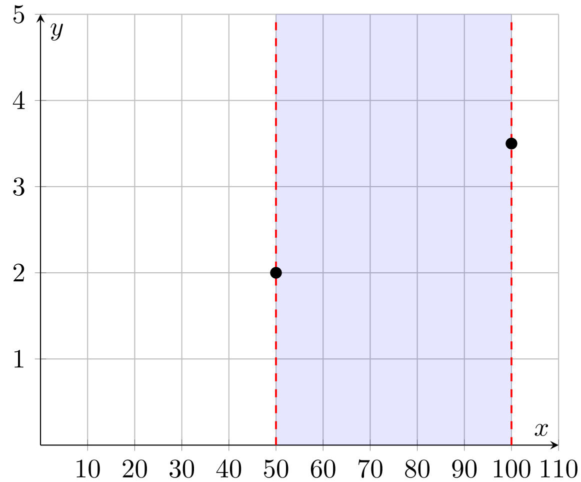
iii. 

A real function f is **strictly increasing** on an interval I (of any form) if:

whenever a and b are real numbers in I and $a < b$,
then $f(a) < f(b)$.

(c) 2.5 points A mysterious real function f is continuous on the interval $[50, 100]$. All we know is that f is continuous and that $f(50) = 2$ and $f(100) = 3.5$.

Therefore, the graph of f could be anywhere in the shaded region below.



i. If f is strictly increasing on the interval $[50, 100]$, show that f satisfies the horizontal line test over the interval $[50, 100]$.

ii. Unlike humans, trees keep growing heavier as they age. If f is the weight of a certain tree in tons as a function of its age in years (so at the age of 50 years, the tree's weight was $f(50) = 2$ tons), show that the tree will weigh 3 tons at *only* one point in its lifetime.