Math 130 04 – A Survey of Calculus

Practice Exam

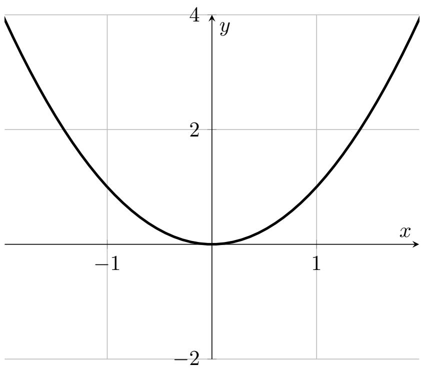
December 8, 2022 **Time: 2 hours**

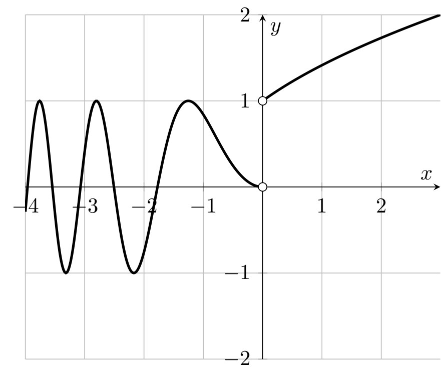
Instructions:

- You have exactly 2 hours to finish the exam.
- You are allowed to use your personal notes (paper only) and a graphing calculator. No other devices (computers, cell phones, tablets) may be used.
- You **must** write your name and student ID at the top of the first page, and you **must** initial every page that you use.
- This exam has **five** questions, each worth **five** points. Your goal is to get **18** points in total.
- Any extra points (> 18) will eventually count towards increasing your grade (A→A$^+$, B$^+$→A, B$^-$→B, and so on) at the end of the semester.
- Each question is divided into subquestions. The points that each subquestion is worth are indicated next to it.
- Write your answers clearly and neatly in the space provided after each question.
- Ask for extra sheets of paper if you need them.
- Number your answers correctly (especially if you're using extra sheets of paper).
- Justify your answers **fully and clearly.** Answers with no explanation (*even if the final calculation is correct*) are worth **zero** points. Answers with a full and correct explanation but a calculation error are worth more than 90% of the points.

Your Name:
Your Student ID:

1. (a) (2 points) Which of the following graphs represent real functions? Which of the functions is continuous over the interval $[-1, 1]$?

i. 

ii. 

Solution:

i. The graph is a real function, since it passes the vertical line test. The function is continuous over the interval $[-1, 1]$, since it passes the pen-to-paper test over this interval.

ii. The graph is a real function, since it passes the vertical line test. The function is not continuous over the interval $[-1, 1]$, since it doesn't pass the pen-to-paper test over this interval (its value at 0 is undefined).

(b) (3 points) Calculate the following limits.

i. $\lim\limits_{x\to 2} \dfrac{3x^2-6}{x^2-3}$ ii. $\lim\limits_{x\to 2} \dfrac{x^4-3x^2-4}{x-2}$ iii. $\lim\limits_{x\to \infty} \dfrac{x^2+4x-3}{x^3-1}$

Solution:

i. By direct substitution, we have

$$\lim_{x\to 2} \frac{3x^2-6}{x^2-3} = \frac{3(2)^2-6}{(2)^2-3} = \frac{12-6}{4-1} = \boxed{6}$$

ii. If we try to use the substitution formula we get $\frac{0}{0}$, which doesn't make sense. But this tells us that the polynomials $p(x) = x^4 - 3x^2 - 4$ and $q(x) = x - 2$ both have 2 as a root (that is, $p(2) = 0$ and $q(2) = 0$). Therefore $p(x) = (x-2) \cdot p_1(x)$ and of course $q(x) = (x-2) \cdot 1$. We can find $p_1(x)$ by long division:

$$\begin{array}{r|l}
 & x^3 + 2x^2 + x + 2 \\
x-2 & x^4 \qquad - 3x^2 \qquad - 4 \\
 & -x^4 + 2x^3 \\ \hline
 & 2x^3 - 3x^2 \\
 & -2x^3 + 4x^2 \\ \hline
 & x^2 \\
 & -x^2 + 2x \\ \hline
 & 2x - 4 \\
 & -2x + 4 \\ \hline
 & 0
\end{array}$$

So we have

$$\begin{aligned}
\lim_{x\to 2} \frac{x^4-3x^2-4}{x-2} &= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^3+2x^2+x+2)}{\cancel{(x-2)} \cdot 1} \\
&= 2^3 + 2(2^2) + 2 + 2 = \boxed{20}.
\end{aligned}$$

iii. If we try to use the substitution formula, we get $\frac{\infty}{\infty}$ which is not a real number (doesn't make sense). Whenever $x > 0$ (which will be the case when $x \to \infty$), we have

$$\begin{aligned}\frac{x^2+4x-3}{x^3-1} &= \frac{\frac{x^2+4x-3}{x^3}}{\frac{x^3-1}{x^3}} \\ &= \frac{\frac{1}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{1-\frac{1}{x^3}}.\end{aligned}$$

So we can use the algebra of limits to get

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{1-\frac{1}{x^3}} &= \frac{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{4}{x^2}-\lim_{x\to\infty}\frac{1}{x^3}}{1-\lim_{x\to\infty}\frac{1}{x^3}} \\ &= \frac{0+0-0}{1-0} = \boxed{0}.\end{aligned}$$

2. (a) (3 points) Calculate the derivatives of the following functions.

i. $f(x) = 6x^{1/3} + 2x^{-3/4}$ ii. $f(x) = \ln(x^2+3)$ iii. $f(x) = e^{(3x^3-\ln(x))}$

Solution:

i. We use the algebra of derivatives to get

$$\frac{d}{dx}\left(6x^{1/3}+2x^{-3/4}\right) = 6\frac{d}{dx}\left(x^{1/3}\right)+2\frac{d}{dx}\left(x^{-3/4}\right) = 6\cdot\frac{1}{3}\cdot x^{-2/3}+2\cdot\left(\frac{-3}{4}\right)\cdot x^{-7/4} = \boxed{2x^{-2/3}-\frac{3}{2}x^{-7/4}}$$

ii. f can be written as the composite function $g \circ h(x) = g(h(x))$, where $h(x) = x^2+3$ and $g(x) = \ln(x)$. Therefore, we can apply the chain rule to get

$$\begin{aligned}f'(x) &= g'(h(x))\cdot h'(x) \\ &= \frac{1}{h(x)}\cdot 2x \\ &= \boxed{\frac{2x}{x^2+3}}\end{aligned}$$

iii. f can be written as the composite function $g \circ h(x) = g(h(x))$, where $g(x) = e^x$ and $h(x) = 3x^3 - \ln(x)$. Therefore, we can apply the chain rule to get

$$\begin{aligned}f'(x) &= g'(h(x))\cdot h'(x) \\ &= e^{h(x)}\cdot(9x^2-\frac{1}{x}) \\ &= \boxed{(9x^2-\frac{1}{x})\cdot e^{(3x^3-\ln(x))}}\end{aligned}$$

(b) (2 points) Calculate the Riemann integrals of the following functions.

i. $f(x) = 4x^3 + 2x^{-2/3}$ ii. $f(x) = e^{4x} + x^{-3/2}$

Solution:

i. We use the rules for Riemann integrals to get

$$\left(\int f\right)(x) = \left(\int_0^x 4x^3 \cdot dx\right) + \left(\int_0^x 2x^{-2/3} \cdot dx\right)$$
$$= 4 \cdot \frac{x^4}{4} + 2 \cdot \frac{x^{1/3}}{\frac{1}{3}}$$
$$= \boxed{x^4 + 6x^{1/3}}$$

ii. We use the rules for Riemann integrals to get

$$\left(\int f\right)(x) = \left(\int_0^x e^{4x} \cdot dx\right) + \left(\int_0^x x^{-3/2} \cdot dx\right)$$
$$= \frac{e^{4x} - 1}{4} + \frac{x^{-1/2}}{-\frac{1}{2}}$$
$$= \boxed{\frac{e^{4x} - 1}{4} - 2x^{-1/2}}$$

3. Consider the following function.

$$f(x) = \begin{cases} x^3 - 6(x^2 + 4)^{1/2} & \text{if } x \geq 0 \\ \\ 4x^2 - 12 & \text{if } x < 0 \end{cases}$$

(a) (1 point) Is f continuous at 0?
(b) (2 points) Is f differentiable at 0?
(c) (2 points) Does f have a local maximum or a local minimum at 0?

Solution:

(a) When $x < 0$, $f(x) = 4x^2 + 12$. So we have

$$\lim_{x \to 0^-} f(x) = 4(0^2) + 12 = 12.$$

When $x > 0$, $f(x) = x^3 - 6(x^2 + 4)^{1/2}$. So we have

$$\lim_{x \to 0^+} f(x) = 0^3 - 6(0^2 + 4)^{1/2} = 6(4^{1/2}) = 12.$$

Finally, when $x = 0$, $f(x) = 12$. Therefore, since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$, $\boxed{f \text{ is continuous at } 0}$.

(b) In order to be differentiable at 0, f must be continuous at 0 and the limit $\lim_{h \to 0} D_h f(0)$ must exist. We have just shown that f is continuous at 0, so now we need to show that $\lim_{h \to 0} D_h f(0)$ exists.

We have

$$\lim_{h \to 0^-} D_h f(0) = \lim_{h \to 0^-} D_h g(0) = g'(0)$$

where $g(x) = 4x^2 - 12$. Since $g'(x) = 8x$, we have $g'(0) = 0$

We have

$$\lim_{h \to 0^+} D_h f(0) = \lim_{h \to 0^+} D_h u(0) = u'(0)$$

where $u(x) = x^3 - 6(x^2 + 4)^{1/2}$. Using the chain rule, we have

$$u'(x) = 3x^2 - 3(x^2 + 4)^{-1/2} \cdot 2x$$

and so $u'(0) = 0$.

Therefore, since $\lim_{h \to 0^-} D_h f(0) = \lim_{h \to 0^+} D_h f(0)$, the derivative of f at 0 exists, and so $\boxed{f \text{ is differentiable at } 0}$.

4. A company estimates their total cost function to produce x units to be

$$C(x) = 4000 + 0.25x^2 \qquad \text{thousand dollars.}$$

The company also estimates that in order to sell x units, each unit must be priced at

$$f(x) = 150 - 0.5x \qquad \text{thousand dollars.}$$

(a) (2 points) Assuming x units are produced and sold, calculate the total revenue function $R(x)$ and the total profit function $P(x)$.

(b) (2 points) How many units must be produced and sold to maximize profit? What is the maximum profit?

(c) (1 point) What price per unit must be charged to maximize profit?

Solution:

(a) The total revenue from selling x units is at $f(x)$ thousand dollars each is

$$R(x) = x \cdot f(x) = 150x - 0.5x^2 \qquad \text{thousand dollars.}$$

The total profit from producing and selling x units is

$$P(x) = R(x) - C(x) = 150x - 0.5x^2 - 4000 - 0.25x^2 = 150x - 0.75x^2 - 4000.$$

(b) To maximize profit, we need to find a maximum of the total profit function P. To do so, we first calculate the critical points of P by finding the derivative

$$P'(x) = 150 - 1.5x$$

and then solving $P'(x) = 0$ to get $x = 100$ as the only critical point of P. To check if $x = 100$ is a maximum of P, we need to check that $P''(100) < 0$. Since $P''(x) = -1.5$, $P''(100) = -1.5$, and so $x = 100$ is a maximum of P.

Therefore, profit will be maximized by producing and selling 100 units. The maximum profit is $P(100) = 3500$ thousand dollars, or 3.5 million dollars.

(c) The price per unit that must be charged to maximize profit is $f(100) = 100$ thousand dollars.

5. Like all mammals, humans' bodies are maintained at a fixed temperature (98.6 degrees Fahrenheit) while they are alive. When a person dies, their corpse's temperature decreases as follows: At x hours after death, the corpse's temperature is

$$T(x) = T_0 + (98.6 - T_0)e^{-kx} \qquad \text{degrees Fahrenheit,}$$

where T_0 is the ambient temperature (of the room or environment) and k is a positive constant real number.

Upon arrival, a coroner finds the temperature of a corpse to be 61.6 degrees Fahrenheit. After 1 hour, the coroner measures the corpse's temperature to be 57.2 degrees Fahrenheit. The corpse is in a location whose ambient temperature is 10 degrees Fahrenheit.

(a) (2 points) If the coroner arrived x hours after the person died, then use the equation $\dfrac{T(x)}{T(x+1)} = \dfrac{61.6}{57.2}$ to find the constant k.

(b) ($1\frac{1}{2}$ points) If the coroner arrived at 11 PM, when did the person die?

(c) ($1\frac{1}{2}$ points) What was the rate of change of the corpse's temperature (in degrees Fahrenheit per hour) when the coroner arrived?

Solution:

(a) We know that $T(x) = 61.6$ degrees Fahrenheit and $T(x+1) = 57.2$ degrees Fahrenheit. Therefore, we have

$$\begin{aligned}
& \frac{T(x)}{T(x+1)} = \frac{61.6}{57.2} \\
\text{i.e.} \quad & \frac{(98.6 - T_0)e^{-kx}}{(98.6 - T_0)e^{-k(x+1)}} = \frac{61.6}{57.2} \\
\text{i.e.} \quad & e^{k(x+1)-kx} = \frac{61.6}{57.2} \\
\text{i.e.} \quad & e^{k} = \frac{61.6}{57.2} \\
\text{i.e.} \quad & k = \ln\left(\frac{61.6}{57.2}\right) \\
\text{i.e.} \quad & k = \ln 61.6 - \ln 57.2 \approx \boxed{0.074}.
\end{aligned}$$

(b) Since $T(x) = 61.6$ and $T_0 = 10$ degrees Fahrenheit, we have

$$(98.6 - T_0)e^{-0.074x} = 61.6$$
$$(98.6 - 10)e^{-.074x} = 61.6$$
$$e^{.074x} = \frac{88.6}{61.6}$$
$$.074x = \ln\left(\frac{88.6}{61.6}\right) x = \frac{1}{.074}\ln\left(\frac{88.6}{61.6}\right) \approx 4.91$$

Therefore the coroner arrived about 5 hours after the person died, i.e. the person died at 6 PM.

(c) The rate of change of the corpse's temperature when the coroner arrived was $T'(4.91)$. Since

$$T'(x) = k(T_0 - T(x))$$

we have $T'(4.91) = .074(10 - 61.6) \approx -3.82$ degrees Fahrenheit per hour.