Math 150 03 – Calculus I

Homework assignment 3

Due: Wednesday, October 4, 2023

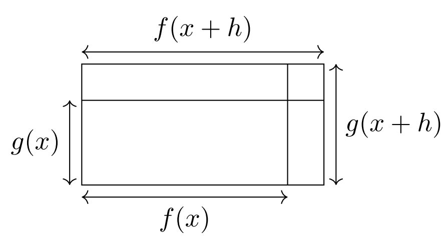
Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
Justify all your answers.

1. **(The product rule for derivatives.)**

Let f and g be real functions that are differentiable at a real number x (i.e. such that the derivatives $f'(x)$ and $g'(x)$ exist). We would like to calculate the derivative $(f \cdot g)'(x)$ of the product function $f \cdot g$. Remember that this function is defined as:

$$f \cdot g(x) = f(x) \cdot g(x)$$

(a) Suppose that h is a small real number. In the figure below, shade the region that corresponds to the value $\big(f \cdot g(x+h) - f \cdot g(x)\big)$.



(b) Use the figure to show that this value can also be written as

$$\Big(\big(f(x+h) - f(x)\big) \cdot g(x+h)\Big) + \Big(f(x) \cdot \big(g(x+h) - g(x)\big)\Big)$$

(c) Use the previous answer to show that the difference quotient $D(f \cdot g)_x(h)$ can be written as

$$D(f \cdot g)_x(h) = \big(Df_x(h) \cdot g(x+h)\big) + \big(f(x) \cdot Dg_x(h)\big)$$

(d) Since g is differentiable at x, g must be continuous at x (we will see why later). Show that if g is continuous at x, then we have that

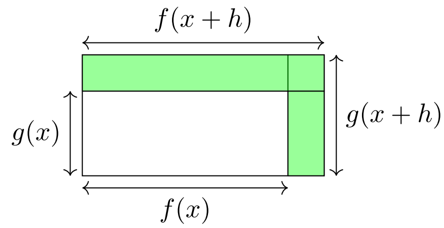
$$\lim_{h \to 0} g(x+h) = g(x)$$

(e) Use the previous two answers and the algebra of limits to show the **product rule for derivatives**:

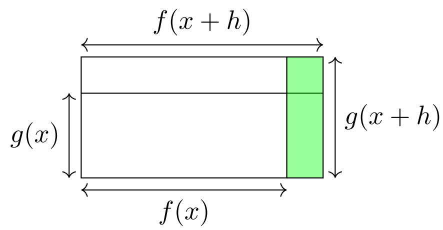
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

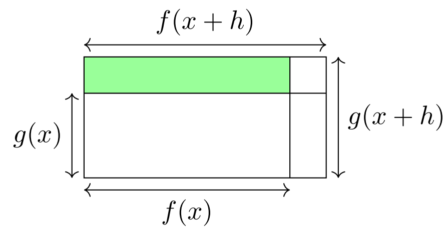
Solution:

(a) Since $f \cdot g(x+h) = f(x+h) \cdot g(x+h)$ is the area of the outer rectangle, and $f \cdot g(x) = f(x) \cdot g(x)$ is the area of the inner rectangle, the shaded region below corresponds to $\boxed{\big(f \cdot g(x+h) - f \cdot g(x)\big)}$.



(b) The previous shaded region is the sum of the two regions below.





The area of the first rectangle is $\big(f(x+h) - f(x)\big) \cdot g(x+h)$ and the area of the second one is $f(x) \cdot \big(g(x+h) - g(x)\big)$. Therefore we have

$$\boxed{\big(f \cdot g(x+h) - f \cdot g(x)\big) = \Big(\big(f(x+h) - f(x)\big) \cdot g(x+h)\Big) + \Big(f(x) \cdot \big(g(x+h) - g(x)\big)\Big)}$$

(c) We have

$$\begin{aligned}
D(f \cdot g)_x(h) &= \frac{f \cdot g(x+h) - f \cdot g(x)}{h} && \text{(by definition of } D(f \cdot g)_x(h)\text{)} \\
&= \frac{\Big(\big(f(x+h) - f(x)\big) \cdot g(x+h)\Big) + \Big(f(x) \cdot \big(g(x+h) - g(x)\big)\Big)}{h} && \text{(from the previous answer)} \\
&= \left(\frac{\big(f(x+h) - f(x)\big)}{h} \cdot g(x+h)\right) + \left(f(x) \cdot \frac{\big(g(x+h) - g(x)\big)}{h}\right) \\
&= \boxed{Df_x(h) \cdot g(x+h) + f(x) \cdot Dg_x(h)}
\end{aligned}$$

(d) Recall the fact about continuous functions and limits:

Fact: If i is any real function such that the limit $\lim_{x \to a} i(x)$ exists, and if j is any real function that is continuous at $\lim_{x \to a} i(x)$, then we have:

$$\lim_{x \to a} j(i(x)) = j\left(\lim_{x \to a} i(x)\right)$$

We can apply this fact to the function $i(h) = x + h$. We know (using the algebra of limits) that $\lim_{h \to 0} i(h) = x$. We also know that g is continuous at x. Therefore the fact tells us that

$$\boxed{\lim_{h \to 0} g(x+h) = \lim_{h \to 0} g(i(h)) = g(\lim_{h \to 0} i(h)) = g(x)}.$$

(e) We have

$$\begin{aligned}
(f\cdot g)'(x) &= \lim_{h\to 0} D(f\cdot g)_x(h) \\
&= \lim_{h\to 0}\Big(Df_x(h)\cdot g(x+h) + f(x)\cdot Dg_x(h)\Big) && \text{(using answer (c))} \\
&= \Big(\lim_{h\to 0} Df_x(h)\Big)\cdot\Big(\lim_{h\to 0} g(x+h)\Big) + f(x)\cdot\Big(\lim_{h\to 0} Dg_x(h)\Big) && \text{(using the algebra of limits)} \\
&= \boxed{f'(x)\cdot g(x) + f(x)\cdot g'(x)} && \text{(using answer (d) and the definition of } f' \text{ and } g')
\end{aligned}$$

2. (a) Let f be the function $f(x) = x^3$. Calculate the h-difference quotient $Df_x(h)$ and the derivative function f' using the limit definition of the derivative.
   (b) Since the function $f(x) = x^3$ can be written as $f(x) = x \cdot x^2$, use the product rule for derivatives to calculate the derivative $f'(x)$. Is this easier than the calculation in the previous question?
   (c) Use the product rule for derivatives to calculate the derivatives of the functions:
      i. $f(x) = x^4$ (Hint: $f(x) = x \cdot x^3$)
      ii. $f(x) = x^5$
      iii. $f(x) = x^6$
   (d) Infer a general formula for the derivative of the function $f(x) = x^n$, where $n \in \mathbb{N}$ is some natural number.
   (e) Use the previous answer to calculate the derivative of the function $f(x) = x^9 - 4x^3 + 7$.

Solution:

(a) We have

$$\begin{aligned}
Df_x(h) &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h} \\
&= \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2}}{\cancel{h}} \\
&= \boxed{3x^2 + 3xh + h^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} Df_x(h) \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&= 3x^2 + 3x\cdot 0 + 0^2 \\
&= \boxed{3x^2}.
\end{aligned}$$

(b) We saw in class that the derivative of the function $g(x) = x$ is $g'(x) = 1$ and the derivative of the function $i(x) = x^2$ is $i'(x) = 2x$. Since we can write $f(x) = x \cdot x^2 = g(x) \cdot i(x)$, we can use the product rule to find:

$$\begin{aligned} f'(x) &= g'(x) \cdot i(x) + g(x) \cdot i'(x) \\ &= x^2 + x \cdot 2x \\ &= \boxed{3x^2}. \end{aligned}$$

(c) i. We have $f(x) = x \cdot x^3 = g(x) \cdot i(x)$. We use the prouct rule to find

$$f'(x) = x^3 + x \cdot 3x^2 = \boxed{4x^3}.$$

ii. We have $f(x) = x \cdot x^4 = g(x) \cdot i(x)$. We use the prouct rule to find

$$f'(x) = x^4 + x \cdot 4x^3 = \boxed{5x^4}.$$

iii. We have $f(x) = x \cdot x^5 = g(x) \cdot i(x)$. We use the prouct rule to find

$$f'(x) = x^5 + x \cdot 5x^4 = \boxed{6x^5}.$$

(d) In general, if $n \in \mathbb{N}$ is some natural number bigger than 6, we can write $f(x) = x^n = x \cdot x^{n-1} = g(x) \cdot i(x)$, and we can suppose that we have found that $i'(x) = (n-1) \cdot x^{n-2}$. So using the product rule, we have

$$f'(x) = x^{n-1} + x \cdot (n-1)x^{n-2} = \boxed{nx^{n-1}}.$$

(e) We have $f(x) = g(x) - 4h(x) + 7$. Therefore

$$\begin{aligned} f'(x) &= g'(x) - 4h'(x) + 0 && \text{(using the sum and constant rules for derivatives)} \\ &= 9x^8 - 4 \cdot 3x^2 && \text{(using the formula we just found)} \\ &= \boxed{9x^8 - 12x^2}. \end{aligned}$$