Math 150 03 – Calculus I

Homework assignment 5

Due: Wednesday, November 1, 2023

1. (a) Find $\frac{d}{dx}[y]$ in terms of x and y if we have that $x \cdot \ln(y) + y^3 = 3 \cdot \ln(x)$.

   (b) Use implicit differentiation to find the tangent line to the curve $x = y^5 - 5y^3 + 4y$ at the point $(0, 1)$.

   (c) Use implicit differentiation to find the tangent line to the curve $\sin(x + y) + \cos(x - y) = 1$ at the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

Solution:

(a) We are assuming that $y(x)$ is an implicit function of x, and the question asks us to find the derivative $y'(x)$ in terms of x and $y(x)$. Differentiating both sides of the given equation, we have:

$$\frac{d}{dx}[3 \cdot \ln(x)] = \frac{d}{dx}\left[x \cdot \ln(y(x)) + y(x)^3\right]$$

$$\text{i.e.} \quad 3 \cdot \frac{1}{x} = \frac{d}{dx}[x \cdot \ln(y(x))] + \frac{d}{dx}\left[y(x)^3\right] \qquad \text{(using the sum and constant multiple rules)}$$

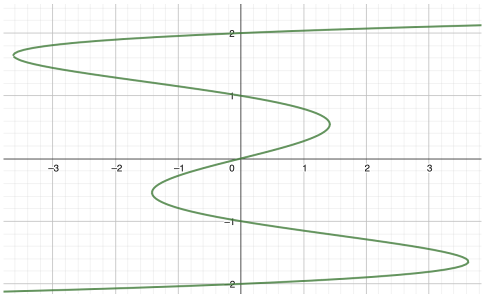
$$\text{i.e.} \quad \frac{3}{x} = \left(1 \cdot \ln(y(x)) + x \cdot \frac{d}{dx}[\ln(y(x))]\right) + 3y(x)^2 \cdot y'(x) \qquad \text{(using product and chain rules}$$

$$\text{i.e.} \quad \frac{3}{x} = \ln(y(x)) + \frac{1}{y(x)} \cdot y'(x) + 3y(x)^2 \cdot y'(x) \qquad \text{(using the chain rule)}$$

$$\text{i.e.} \quad \frac{3}{x} - \ln(y(x)) = y'(x) \cdot \left(\frac{1}{y(x)} + 3y(x)^2\right)$$

$$\text{i.e.} \quad \frac{d}{dx}[y] = \boxed{\frac{\frac{3}{x} - \ln(y)}{\frac{1}{y} + 3y^2}}.$$

(b) The implicit function theorem tells us that if we zoom into a small neighborhood of the point $(0, 1)$, then the curve given by $x = y^5 - 5y^3 + 4y$ looks like a function. (That is, it satisfies the vertical line test.) We can verify this using a graphing calculator (like Desmos or Geogebra). Below on the left is the graph of the curve, and on the right is the same graph but zoomed into a neighborhood of the point $(0, 1)$.



Therefore, we can implicitly assume y to be some function $y(x)$ (in the neighborhood of the point $(0,1)$) and differentiate both sides of the equation of the curve to get

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[y(x)^5 - 5y(x)^3 + 4y(x)\right]$$

$$\text{i.e.} \quad 1 = 5y(x)^4 y'(x) - 15y(x)^2 y'(x) + 4y'(x) \qquad \text{(using the sum and chain rules)}$$

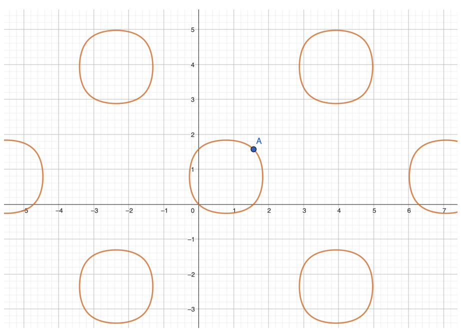
$$\text{i.e.} \quad y'(x) = \frac{1}{5y(x)^4 - 15y(x)^2 + 4}$$

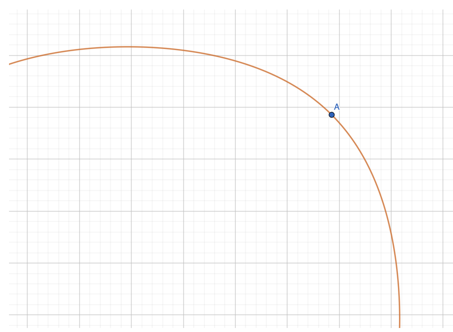
Since at the point $(0,1)$, we have $x = 0$ and $y(0) = 1$, we have

$$\begin{aligned} y'(0) &= \frac{1}{5y(0)^4 - 15y(0)^2 + 4} \\ &= \frac{1}{1 - 15 + 4} \\ &= -\frac{1}{10} = -0.1 \end{aligned}$$

Therefore, the slope of the tangent line to the curve at the point $(0,1)$ is $y'(0) = -0.1$. Since we know the slope of the line and a point on it (namely $(0,1)$), the equation of the line (in point-slope form) is $\boxed{y - 1 = -0.1(x - 0)}$. In standard form, the equation of the tangent line is $\boxed{y = -0.1x + 1}$.

(c) The implicit function theorem tells us that if we zoom into a small neighborhood of the point $(\frac{\pi}{2}, \frac{\pi}{2})$, then the curve given by $\sin(x+y) + \cos(x-y) = 1$ looks like a function. (That is, it satisfies the vertical line test.) We can verify this using a graphing calculator (like Desmos or Geogebra). Below on the left is the graph of the curve, and on the right is the same graph but zoomed into a neighborhood of the point $(\frac{\pi}{2}, \frac{\pi}{2})$.





Therefore, we can implicitly assume y to be some function $y(x)$ (in the neighborhood of the point $(\frac{\pi}{2}, \frac{\pi}{2})$) and differentiate both sides of the equation of the curve to get

$$\frac{d}{dx}\left[\sin(x + y(x)) + \cos(x - y(x))\right] = \frac{d}{dx}[1]$$

$$\text{i.e.} \quad \cos(x + y(x))\frac{d}{dx}[x + y(x)] - \sin(x - y(x))\frac{d}{dx}[x - y(x)] = 0 \qquad \text{(using the sum and chain rules)}$$

$$\text{i.e.} \quad \cos(x + y(x))(1 + y'(x)) - \sin(x - y(x))(1 - y'(x)) = 0 \qquad \text{(using the sum rule)}$$

$$\text{i.e.} \quad y'(x)(\cos(x + y(x)) + \sin(x - y(x))) = \sin(x - y(x)) - \cos(x + y(x))$$

$$\text{i.e.} \quad y'(x) = \frac{\sin(x - y(x)) - \cos(x + y(x))}{(\cos(x + y(x)) + \sin(x - y(x)))}$$

Since at the point $(\frac{\pi}{2}, \frac{\pi}{2})$, we have $x = \frac{\pi}{2}$ and $y(\frac{\pi}{2}) = \frac{\pi}{2}$, we have

$$\begin{aligned} y'\left(\frac{\pi}{2}\right) &= \frac{\sin(\frac{\pi}{2} - \frac{\pi}{2}) - \cos(\frac{\pi}{2} + \frac{\pi}{2})}{(\cos(\frac{\pi}{2} + \frac{\pi}{2}) + \sin(\frac{\pi}{2} - \frac{\pi}{2})} \\ &= \frac{\sin(0) - \cos(\pi)}{\cos(\pi) + \sin(0)} \\ &= \frac{0 - (-1)}{(-1) + 0} = -1 \end{aligned}$$

Therefore, the slope of the tangent line to the curve at the point $(\frac{\pi}{2}, \frac{\pi}{2})$ is $y'(\frac{\pi}{2}) = -1$. Since we know the slope of the line and a point on it (namely $(\frac{\pi}{2}, \frac{\pi}{2})$), the equation of the line (in point-slope form) is $\boxed{y - \frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)}$. In standard form, the equation of the tangent line is $\boxed{y = -x + \pi}$.

2. Use L'Hôpital's rule where appropriate to find the following limits.

(a) $\lim_{x \to 4} \frac{\ln(\frac{x}{4})}{x^2 - 16}$

(b) $\lim_{x \to 0} \frac{1 - \cos(7x)}{1 - \cos(3x)}$

(c) $\lim_{x \to 1} \frac{4^x - 3^x - 1}{x^2 - 1}$

Solution:

(a) We proceed by steps.

1. By direct substitution, we have that

$$\lim_{x \to 4} \frac{\ln(\frac{x}{4})}{x^2 - 16} = \frac{\ln(\frac{4}{4})}{4^2 - 16} = \frac{0}{0}$$

which is a nonsense answer (an indeterminate form).

2. Next, if $f(x) = \ln(\frac{x}{4}) = \ln(x) - \ln(4)$ (the numerator of our expression), then $f'(x) = \frac{1}{x}$. If $g(x) = x^2 - 16$ (the denominator of our expression), we have that $g'(x) = 2x$. Therefore, both derivatives exist.

3. We can verify that $g'(x) = 2x \neq 0$ whenever $x \in (4-h, 4) \cup (4, 4+h)$ for some small $h > 0$ (i.e. whenever x is close to, but not equal to 4).

4. Therefore we can apply l'Hôpital's rule to $\frac{f(x)}{g(x)}$, which tells us that

$$\lim_{x \to 4} \frac{f(x)}{g(x)} = \lim_{x \to 4} \frac{f'(x)}{g'(x)} = \lim_{x \to 4} \frac{\frac{1}{x}}{2x} = \lim_{x \to 4} \frac{1}{2x^2} = \frac{1}{2(4^2)} = \boxed{\frac{1}{32}}$$

(b) We proceed by steps.

1. By direct substitution,

$$\lim_{x \to 0} \frac{1 - \cos(7x)}{1 - \cos(3x)} = \frac{1 - \cos(0)}{1 - \cos(0)} = \frac{0}{0}$$

which is a nonsense answer (an indeterminate form).

2. Next, if $f(x) = 1 - \cos(7x)$ (the numerator of our expression), then $f'(x) = 7\sin(7x)$ (using the chain rule). If $g(x) = 1 - \cos(3x)$ (the denominator of our expression), we have that $g'(x) = 3\sin(3x)$ (also using the chain rule). Therefore, both derivatives exist.
3. We can verify that $g'(x) = 3\sin(3x) \neq 0$ whenever $x \in (-h, 0) \cup (0, h)$ for some small $h > 0$ (i.e. whenever x is close to, but not equal to 0).
4. Therefore we can apply l'Hôpital's rule to $\frac{f(x)}{g(x)}$, which tells us that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{7\sin(7x)}{3\sin(3x)}$$

5. If we try direct substitution, we get $\frac{7\sin(0)}{3\sin(0)} = \frac{0}{0}$, which is once again a nonsense answer.
6. We can repeat the process, by calculating $f''(x) = 49\cos(7x)$ (once again using the chain rule), and $g''(x) = 9\cos(3x)$ (also using the chain rule).
7. We can verify that $g''(x) = 9\cos(3x) \neq 0$ whenever $x \in (-h, 0) \cup (0, h)$ for some small $h > 0$ (i.e. whenever x is close to, but not equal to 0).
8. Therefore we can reapply l'Hôpital's rule to $\frac{f'(x)}{g'(x)}$, which tells us that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{49\cos(7x)}{9\cos(3x)} = \frac{49\cos(0)}{9\cos(0)} = \boxed{\frac{49}{9}}$$

(c) We proceed by steps.

1. By direct substitution,

$$\lim_{x\to 1} \frac{4^x - 3^x - 1}{x^2 - 1} = \frac{4^1 - 3^1 - 1}{1^2 - 1} = \frac{0}{0}$$

which is a nonsense answer (an indeterminate form).
2. Next, if $f(x) = 4^x - 3^x - 1$ (the numerator of our expression), then $f'(x) = 4^x\ln(4) - 3^x\ln(3)$ (using the rule for exponential functions). If $g(x) = x^2 - 1$ (the denominator of our expression), we have that $g'(x) = 2x$. Therefore, both derivatives exist.
3. We can verify that $g'(x) = 2x \neq 0$ whenever $x \in (1-h, 1) \cup (1, 1+h)$ for some small $h > 0$ (i.e. whenever x is close to, but not equal to 1).
4. Therefore we can apply l'Hôpital's rule to $\frac{f(x)}{g(x)}$, which tells us that

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{4^x\ln(4) - 3^x\ln(3)}{2x} = \boxed{\frac{4\ln(4) - 3\ln(3)}{2}}$$

(L'Hôpital's rule at $\pm\infty$)
When $x \to a$ (where a is any real number or $\pm\infty$), L'Hôpital's rule states that if $f(x)$ and $g(x)$ both approach 0 or both approach $\pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the right hand limit exists, and provided $g'(x) \neq 0$ whenever $x \in (a-h, a) \cup (a, a+h)$ for some $h > 0$ (or, if $a = \pm\infty$, whenever $x \in (h, \infty)$ or $(-\infty, h)$ as the case may be).

3. Evaluate the following limits, using L'Hôpital's rule as appropriate.

   (a) $\lim_{x\to\infty} \frac{15x^3}{e^{2x}}$

   (b) $\lim_{x\to\infty} \frac{e^x + x}{e^x + x^2}$

Solution:

(a) We proceed by steps.

1. By direct substitution,
$$\lim_{x\to\infty} \frac{15x^3}{e^{2x}} = \frac{\infty}{\infty}$$
which is a nonsense answer (an indeterminate form).

2. Next, if $f(x) = 15x^3$ (the numerator of our expression), then $f'(x) = 45x^2$ (using the rule for exponential functions). If $g(x) = e^{2x}$ (the denominator of our expression), we have that $g'(x) = 2e^{2x}$ (using the chain rule). Therefore, both derivatives exist.

3. We can verify that $g'(x) = 2e^{2x} \neq 0$ whenever $x \in (h, \infty)$ for some large enough $h > 0$ (i.e. whenever x is bigger than some number h).

4. Therefore we can apply l'Hôpital's rule to $\frac{f(x)}{g(x)}$, which tells us that
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{45x^2}{2e^{2x}}$$

5. By direct substitution we get $\frac{\infty}{\infty}$, which is once again a nonsense answer (an indeterminate form).

6. We can repeat the process by calculating $f''(x) = 90x$ and $g''(x) = 4e^{2x}$.

7. Since $g''(x) = 4e^{2x} \neq 0$ for all $x \in (h, \infty)$ for some large enough $h > 0$, we can apply l'Hôpital's rule to get
$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \lim_{x\to\infty} \frac{90x}{4e^{2x}}$$

8. By direct substitution we get $\frac{\infty}{\infty}$, which is once again a nonsense answer (an indeterminate form).

9. We can repeat the process by calculating $f'''(x) = 90$ and $g'''(x) = 8e^{2x}$.

10. Since $g'''(x) = 4e^{2x} \neq 0$ for all $x \in (h, \infty)$ for some large enough $h > 0$, we can apply l'Hôpital's rule to get
$$\lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \lim_{x\to\infty} \frac{f'''(x)}{g'''(x)} = \lim_{x\to\infty} \frac{90}{8e^{2x}} = \boxed{0}$$
since 90 is a constant, and since $e^{2x} \to \infty$ as $x \to \infty$.

(b) We proceed by steps.

1. By direct substitution,
$$\lim_{x\to\infty} \frac{e^x + x}{e^x + x^2} = \frac{\infty}{\infty}$$
which is a nonsense answer (an indeterminate form).

2. Next, if $f(x) = e^x + x$ (the numerator of our expression), then $f'(x) = e^x + 1$. If $g(x) = e^x + x^2$ (the denominator of our expression), we have that $g'(x) = e^x + 2x$. Therefore, both derivatives exist.
3. We can verify that $g'(x) = e^x + 2x \neq 0$ whenever $x \in (h, \infty)$ for some large enough $h > 0$ (i.e. whenever x is bigger than some number h).
4. Therefore we can apply l'Hôpital's rule to $\frac{f(x)}{g(x)}$, which tells us that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{e^x + 1}{e^x + 2x}$$

5. By direct substitution we get $\frac{\infty}{\infty}$, which is once again a nonsense answer (an indeterminate form).
6. We can repeat the process by calculating $f''(x) = e^x$ and $g''(x) = e^x + 2$.
7. Since $g''(x) = e^x + 2 \neq 0$ for all $x \in (h, \infty)$ for some large enough $h > 0$, we can apply l'Hôpital's rule to get

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \lim_{x\to\infty} \frac{e^x}{e^x + 2}$$

8. By direct substitution we get $\frac{\infty}{\infty}$, which is once again a nonsense answer (an indeterminate form).
9. We can repeat the process by calculating $f'''(x) = e^x$ and $g'''(x) = e^x$.
10. Since $g'''(x) = e^x \neq 0$ for all $x \in (h, \infty)$ for some large enough $h > 0$, we can apply l'Hôpital's rule to get

$$\lim_{x\to\infty} \frac{f''(x)}{g''(x)} = \lim_{x\to\infty} \frac{f'''(x)}{g'''(x)} = \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = \boxed{1}.$$

4. We say that a function g **dominates** a function f when we have $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(a) Which function dominates the other: $\ln(x)$ or $\sqrt{x}$?

(b) Which function dominates the other: $\ln(x)$ or $x^{1/n}$? (n is any natural number bigger than 1)

(c) Explain why e^x dominates every polynomial.

Solution:

(a) We know that $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim_{x\to\infty} \sqrt{x} = \infty$. We need to figure out if $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = 0$ or if $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln(x)} = 0$.

To find $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$, we proceed by steps.

1. Direct substitution gives us $\frac{\infty}{\infty}$, which is a nonsense answer.
2. We can calculate $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ and $\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$. Therefore the derivatives of both the numerator and denominator exist.
3. We can verify that $\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \neq 0$ when $x \in (h, \infty)$ for some large enough $h > 0$. Therefore we can apply l'Hôpital's rule.

4. Applying l'Hôpital's rule to $\frac{\ln(x)}{\sqrt{x}}$, we get

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

since 2 is a constant and since $\sqrt{x} \to \infty$ as $x \to \infty$.

Therefore $\boxed{\sqrt{x} \text{ dominates } \ln(x).}$

(b) We know that $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim_{x\to\infty} x^{\frac{1}{n}} = \infty$. We need to figure out if $\lim_{x\to\infty} \frac{\ln(x)}{x^{\frac{1}{n}}} = 0$ or if $\lim_{x\to\infty} \frac{x^{\frac{1}{n}}}{\ln(x)} = 0$.

To find $\lim_{x\to\infty} \frac{\ln(x)}{x^{\frac{1}{n}}}$, we proceed by steps.

1. Direct substitution gives us $\frac{\infty}{\infty}$, which is a nonsense answer.
2. We can calculate $\frac{d}{dx}[\ln(x)] = \frac{1}{x} = x^{-1}$ and $\frac{d}{dx}\left[x^{\frac{1}{n}}\right] = \frac{1}{n}x^{\frac{1}{n}-1}$. Therefore the derivatives of both the numerator and denominator exist.
3. We can verify that $\frac{d}{dx}\left[x^{\frac{1}{n}}\right] = \frac{1}{n}x^{\frac{1}{n}-1} \neq 0$ when $x \in (h, \infty)$ for some large enough $h > 0$. Therefore we can apply l'Hôpital's rule.
4. Applying l'Hôpital's rule to $\frac{\ln(x)}{x^{\frac{1}{n}}}$, we get

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{\frac{1}{n}}} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{n}x^{\frac{1}{n}-1}} = \lim_{x\to\infty} \frac{n}{x^{\frac{1}{n}-1+1}} = \lim_{x\to\infty} \frac{n}{x^{\frac{1}{n}}} = 0$$

since n is a constant and since $x^{\frac{1}{n}} \to \infty$ as $x \to \infty$.

Therefore $\boxed{x^{\frac{1}{n}} \text{ dominates } \ln(x).}$

(c) Any polynomial function is of the form $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are all constants. When we differentiate p, we get $p'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + n \cdot a_nx^{n-1}$. Notice that the constant a_0 has disappeared. Therefore, when $p(x)$ is differentiated $n+1$ times, every term disappears and we get $p^{(n+1)}(x) = 0$ (i.e. the $(n+1)$'th derivative of p is 0).

Now if $f(x) = e^x$, we know that $f^{(n+1)}(x) = e^x$ (the $(n+1)$'th derivative of e^x is always e^x). Therefore applying l'Hôpital's rule $(n+1)$ times to $\frac{p(x)}{e^x}$ (where $p(x)$ is any polynomial) gives us $\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$. Therefore e^x dominates every polynomial.