Math 150 03 – Calculus I

Homework assignment 7

Due: Wednesday, November 15, 2023

Recall the useful form of the Fundamental Theorem of Calculus:

FTC(2): If $\left(\int f\right)$ is any antiderivative of f, then we have that

$$\int_a^b f(x)\cdot dx = \left(\int f\right)(b) - \left(\int f\right)(a)$$

1. Calculate the following definite integrals by first finding an antiderivative of the given function and then using the FTC(2).

   (a) $\int_1^2 [x\cdot \ln(x)]\cdot dx$

   (b) $\int_0^1 \left[e^{-x}\cdot (2e^{-x}+3)^9\right]\cdot dx$

   (c) $\int_0^{\frac{\pi}{4}} \left[x\cdot (\sec(x))^2\right]\cdot dx$

Solution:

(a)

1. First, we find an antiderivative of $h(x) = x\cdot\ln(x)$. We have

$$h(x) = x\cdot \ln(x) = \ln(x)\cdot x = f(x)\cdot g'(x)$$

   where $f(x)=\ln(x)$ and $g'(x) = x$.

2. Therefore, we can use the anti-product rule, which states that if $h(x) = f(x)\cdot g'(x)$, then $\left(\int h\right)(x) = f(x)\cdot g(x) - \left(\int (f'\cdot g)\right)(x)$.

3. We have $f'(x) = \frac{d}{dx}[\ln(x)] = \frac{1}{x}$. Since $g'(x) = x$, we have $g(x) = \frac{x^2}{2}$. So we have

$$(f'\cdot g)(x) = f'(x)\cdot g(x) = \frac{1}{x}\cdot\frac{x^2}{2} = \frac{x}{2}$$

   Therefore, using the anti-constant multiple rule and the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1}+C$ (when $a\neq -1$), we have

$$\left(\int (f'\cdot g)\right)(x) = \frac{x^2}{4}+C$$

4. Therefore we have

$$\begin{aligned}\left(\int h\right)(x) &= f(x)\cdot g(x) - \left(\int (f'\cdot g)\right)(x)\\ &= \ln(x)\cdot\frac{x^2}{2} - \frac{x^2}{4} + C\\ &= \frac{x^2}{2}\left(\ln(x)-\frac{1}{2}\right)+C\end{aligned}$$

5. Therefore we have

$$
\begin{aligned}
\int_1^2 h(x) \cdot dx &= \left(\textstyle\int h\right)(2) - \left(\textstyle\int h\right)(1) \\
&= \left[\frac{2^2}{2}\left(\ln(2) - \frac{1}{2}\right) + C\right] - \left[\frac{1^2}{2}\left(\ln(1) - \frac{1}{2}\right) + C\right] \\
&= 2 \cdot \left(\ln(2) - \frac{1}{2}\right) - \frac{1}{2}\left(0 - \frac{1}{2}\right) \\
&= 2\ln(2) - 1 + \frac{1}{4} \\
&= \boxed{2\ln(2) - \frac{3}{4}} \approx 0.63629
\end{aligned}
$$

(b) 1. First, we find an antiderivative of $h(x) = e^{-x} \cdot (2e^{-x} + 3)^9$. We have

$$h(x) = e^{-x} \cdot (2e^{-x} + 3)^9 = (2e^{-x} + 3)^9 \cdot e^{-x} = g(f(x)) \cdot f'(x)$$

where $f(x) = e^{-x}$ (since the "inside" function is e^{-x}).

2. Therefore, we can use the anti-chain rule, which states that $\left(\int h\right)(x) = \left(\int g\right)(f(x))$.

3. Since $f(x) = e^{-x}$, we have $f'(x) = -e^{-x}$ (using the chain rule). Therefore we can solve for $g(x)$:

$$
\begin{aligned}
& g(f(x)) \cdot f'(x) = h(x) \\
\text{i.e.} \quad & g(e^{-x}) \cdot (-e^{-x}) = e^{-x} \cdot (2e^{-x} + 3)^9 \\
\text{i.e.} \quad & g(e^{-x}) = -(2e^{-x} + 3)^9 \qquad \text{(dividing both sides by } -e^{-x}\text{)} \\
\text{i.e.} \quad & g(x) = -(2x + 3)^9
\end{aligned}
$$

4. To find $\left(\int g\right)(x)$, we use the anti-chain rule once again. We need to find $i(x)$ and $j(x)$ such that $g(x) = i(j(x)) \cdot j'(x)$. We can let $j(x) = 2x + 3$ (since that is the "inside" function). Therefore $j'(x) = 2$, and we can solve for $i(x)$:

$$
\begin{aligned}
& i(j(x)) \cdot j'(x) = g(x) \\
\text{i.e.} \quad & i(2x + 3) \cdot 2 = -(2x + 3)^9 \\
\text{i.e.} \quad & i(2x + 3) = -\frac{1}{2}(2x + 3)^9 \\
\text{i.e.} \quad & i(x) = -\frac{1}{2}x^9
\end{aligned}
$$

Therefore we have $\left(\int i\right)(x) = -\frac{1}{2} \cdot \frac{x^{10}}{10}$ and we can find $\left(\int g\right)(x)$:

$$\left(\textstyle\int g\right)(x) = \left(\textstyle\int i\right)(j(x)) = -\frac{1}{20}(2x + 3)^{10} + C$$

5. Now we can find $\left(\int h\right)(x)$:

$$\left(\textstyle\int h\right)(x) = \left(\textstyle\int g\right)(f(x)) = -\frac{1}{20}(2e^{-x} + 3)^{10} + C$$

6. Therefore we have

$$\begin{aligned}
\int_0^1 h(x)\cdot dx &= \left(\textstyle\int h\right)(1) - \left(\textstyle\int h\right)(0)\\
&= \left[-\frac{1}{20}(2e^{-1}+3)^{10}+C\right] - \left[-\frac{1}{20}(2e^{-0}+3)^{10}+C\right]\\
&= \frac{1}{20}(2+3)^{10} - \frac{1}{20}(2e^{-1}+3)^{10}\\
&= \boxed{\frac{1}{20}\left(5^{10}-(2e^{-1}+3)^{10}\right)} \approx 461810.9793
\end{aligned}$$

(c) 1. First, we find an antiderivative of $h(x) = x\cdot(\sec(x))^2$. We have

$$h(x) = x\cdot(\sec(x))^2 = f(x)\cdot g'(x)$$

where $f(x) = x$ and $g'(x) = \sec(x)^2$.

2. Therefore, we can use the anti-product rule, which states that if $h(x) = f(x)\cdot g'(x)$, then $\left(\int h\right)(x) = f(x)\cdot g(x) - \left(\int (f'\cdot g)\right)(x)$.

3. We have $f'(x) = \frac{d}{dx}[x] = 1$. Since $g'(x) = \sec(x)^2$, we have $g(x) = \tan(x)$. So we have

$$(f'\cdot g)(x) = f'(x)\cdot g(x) = \tan(x)$$

Therefore, using the the fact that an antiderivative of $\tan(x)$ is $-\ln(\cos(x)) + C$ (see HW assignment 6), we have

$$\left(\textstyle\int (f'\cdot g)\right)(x) = -\ln(x) + C$$

4. Therefore we have

$$\begin{aligned}
\left(\textstyle\int h\right)(x) &= f(x)\cdot g(x) - \left(\textstyle\int (f'\cdot g)\right)(x)\\
&= x\cdot\tan(x) - (-\ln(\cos(x)) + C\\
&= x\cdot\tan(x) + \ln(\cos(x)) + C
\end{aligned}$$

5. Therefore we have

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} h(x)\cdot dx &= \left(\textstyle\int h\right)(\frac{\pi}{4}) - \left(\textstyle\int h\right)(0)\\
&= \left[\frac{\pi}{4}\cdot\tan\left(\frac{\pi}{4}\right) + \ln(\cos\left(\frac{\pi}{4}\right)) + C\right] - [0\cdot\tan(0) + \ln(\cos(0)) + C]\\
&= \frac{\pi}{4}\cdot 1 + \ln\left(\frac{1}{\sqrt{2}}\right) - (0\cdot 0) + \ln(1)\\
&= \boxed{\frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right)} \approx 0.43882
\end{aligned}$$

2. For each pair of curves given below, find two points where they intersect and calculate the area of the region bounded by them. (Hint: you can use a graphing calculator to visualise the curves, but you should find their points of intersection using a calculation.)

(a) $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{4}x$.

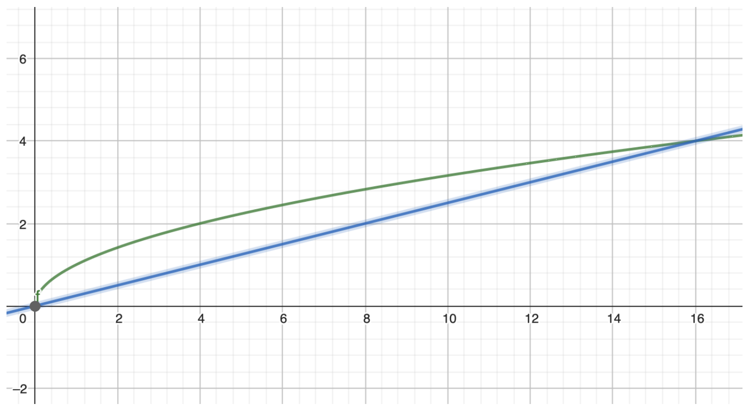
(b) $f(x) = 12 - 2x^2$ and $g(x) = x^2$.

Solution:

(a) To find points of intersection of the two curves, we solve $\sqrt{x} = \frac{1}{4}x$ for x. One solution is $x = 0$, and to find another solution, we have:

$$\begin{aligned} \frac{1}{4}x &= \sqrt{x} \\ \text{i.e.} \quad \sqrt{x} &= 4 \qquad \left(\text{multiplying both sides by } \frac{4}{\sqrt{x}} \text{ since } x \neq 0\right) \\ \text{i.e.} \quad x &= 16 \end{aligned}$$

Therefore, since $f(0) = g(0) = 0$ and $f(16) = g(16) = 4$, the two points of intersection are $(0, 0)$ and $(16, 4)$. We can confirm this by plotting f and g on a graphing calculator.



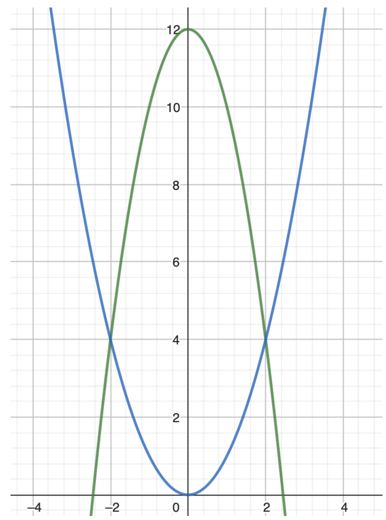
We can see that for every x in the interval $[-2, 2]$, we have $f(x) \geq g(x)$. Therefore the area between the two graphs is given by

$$\begin{aligned} &\int_0^4 [f(t) - g(t)] \cdot dt \\ &= \int_0^4 \left[\sqrt{t} - \frac{1}{4}t\right] \cdot dt \\ &= \int_0^4 t^{1/2} \cdot dt - \frac{1}{4}\int_0^4 t \cdot dt \qquad \text{(using the sum and constant multiple rules)} \\ &= \left(\frac{4^{3/2}}{3/2} - \frac{0^{3/2}}{3/2}\right) - \frac{1}{4}\left(\frac{4^2}{2} - \frac{0^2}{2}\right) \qquad \left(\text{since an antiderivative of } x^a \text{ is } \frac{x^{a+1}}{a+1} + C \text{ when } a \neq -1\right) \\ &= \frac{16}{3} - 2 \\ &= \boxed{\frac{10}{3}} \approx 3.33333 \end{aligned}$$

(b) To find points of intersection of the two curves, we solve $12 - 2x^2 = x^2$ for x. We have:

$$\begin{aligned} 12 - 2x^2 &= x^2 \\ \text{i.e.} \quad 3x^2 &= 12 \\ \text{i.e.} \quad x &= 2 \qquad \text{or} \qquad x = -2 \end{aligned}$$

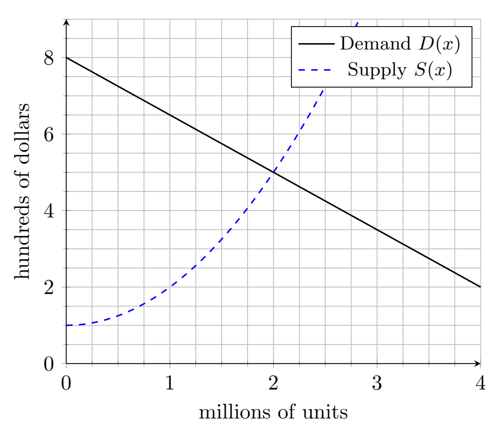
Therefore, the two solutions are $x = 2$ and $x = -2$. Since $f(-2) = g(-2) = 4$ and $f(2) = g(2) = 4$, the two points of intersection are $(-2, 4)$ and $(2, 4)$. We can confirm this by plotting f and g on a graphing calculator.



We can see that for every x in the interval $[0, 4]$, we have $f(x) \geq g(x)$. Therefore the area between the two graphs is given by

$$\int_{-2}^{2} [f(t) - g(t)] \cdot dt$$
$$= \int_{-2}^{2} \left[(12 - 2t^2) - t^2\right] \cdot dt$$
$$= \int_{-2}^{2} 12 \cdot dt - 3\int_{-2}^{2} t^2 \cdot dt \qquad \text{(using the sum and constant multiple rules)}$$
$$= (12(2) - 12(-2)) - 3\left(\frac{2^3}{3} - \frac{(-2)^3}{3}\right) \qquad \text{(since an antiderivative of } x^a \text{ is } \frac{x^{a+1}}{a+1} + C \text{ when } a \neq -1)$$
$$= 48 - 3\left(\frac{16}{3}\right)$$
$$= \boxed{32}$$

3. (Supply and demand)

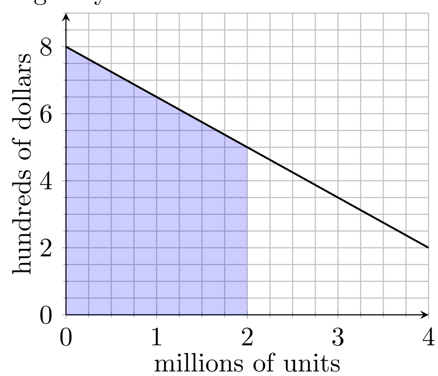


In economics, the *demand curve* of a product is a function $D(x)$ where x is the total number of products that consumers are willing to buy at a price point of $D(x)$ dollars per unit of that product. In general,

D is a decreasing function (since consumers will usually buy more of a product only if the price per unit is less).

Conversely, the *supply curve* of a product is a function $S(x)$ where x is the total number of products that suppliers are willing to sell at a price point of $S(x)$ dollars per unit of that product. In general, S is an increasing function (since suppliers will usually supply more of a product only if the price per unit is higher).

(a) The demand curve for espresso machines is given by $D(x) = 8 - \frac{3}{2}x$ hundred dollars per unit, assuming x million units are bought by consumers.



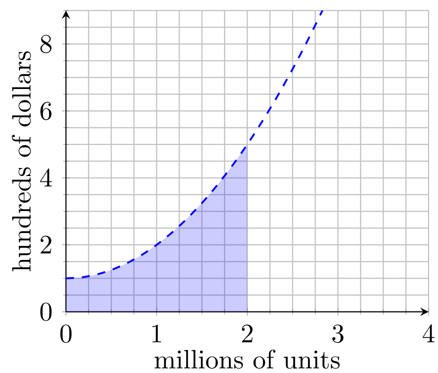
Explain why the area of the shaded region is the *maximum* total amount of money that consumers are willing to spend to buy 2 million espresso machines. What is this amount?

Solution: The demand curve D tells us that if x million espresso machines have already been sold, then the next espresso machine can be sold at a maximum price of $D(x)$ hundred dollars. Therefore, if *every* espresso machine is sold at the maximum price that someone is willing to pay for it, then the shaded region of the graph corresponds to the maximum total amount of money that consumers are willing to spend to buy 2 million espresso machines.

This amount can be calculated as a definite integral:

$$\begin{aligned}\int_0^2 \left[8 - \frac{3}{2}t\right] \cdot dt &= \int_0^2 8 \cdot dt - \frac{3}{2}\int_0^2 t \cdot dt \\ &= [8(2) - 8(0)] - \frac{3}{2}\left[\frac{2^2}{2} - \frac{0^2}{2}\right] \\ &= 16 - \frac{3}{2} \cdot 2 \\ &= 16 - 3 = \boxed{\text{13 hundred million dollars.}}\end{aligned}$$

(b) The supply curve for espresso machines is given by $S(x) = 1+x^2$ hundred dollars per unit, assuming x million units are sold by suppliers.



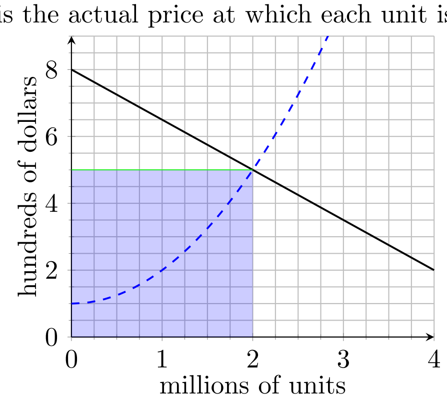
Explain why the area of the shaded region is the *minimum* total amount of money that suppliers are willing to sell 2 million espresso machines for. What is this amount?

Solution: The supply curve S tells us that if x million espresso machines have already been sold, then the next espresso machine can be sold at a minimum price of $S(x)$ hundred dollars. Therefore, if *every* espresso machine is sold at the minimum price that suppliers are willing to sell it for, then the shaded region corresponds the minimum total amount of money that suppliers are willing to sell 2 million espresso machines for.

This amount can be calculated as a definite integral:

$$\begin{aligned}\int_0^2 \left[1+t^2\right] \cdot dt &= \int_0^2 1 \cdot dt + \int_0^2 t^2 \cdot dt \\ &= [1(2) - 1(0)] + \left[\frac{2^3}{3} - \frac{0^3}{3}\right] \\ &= 1 + \frac{8}{3} \\ &= \frac{11}{3} = \boxed{3.6667 \text{ hundred million dollars.}}\end{aligned}$$

(c) The point where the supply and demand curves of a product intersect is called the *market equilibrium.* The x-coordinate of this point represents the number of units that are bought and sold in reality, and the y-coordinate is the actual price at which each unit is sold for.



i. Use the figure to find the market equilibrium for espresso machines. How many espresso machines are bought and sold in reality, and what is the final price per unit of each one?

ii. Explain why the rectangular shaded region in the figure represents the total amount of money given by consumers to suppliers if the number of espresso machines bought and sold, as well as the price of each espresso machine, are given by the market equilibrium.

iii. The difference between the area calculated in question (a) and the area of the rectangular region is called the *consumer surplus*. Calculate this amount for espresso machines. Explain why this is the amount of money that consumers *save* if the number of espresso machines bought and sold, as well as the price of each machine, is determined by the market equilibrium.

iv. The difference between the area of the rectangular region and the area calculated in question (b) is called the *supplier surplus*. Calculate this amount for espresso machines. Explain why this is the amount of money that suppliers *gain* if the number of espresso machines bought and sold, as well as the price of each machine, is determined by the market equilibrium.

Solution:

i. From the figure, we can see that the market equilibrium is the point $(2, 5)$. Since the x-coordinate is 2, therefore 2 million espresso machines are bought and sold in reality. Since the y-coordinate is 5, each of these espresso machines is sold at a price of 5 hundred dollars.

ii. If 2 million espresso machines are bought and sold for a price of 5 hundred dollars each, then the total amount of money given by consumers to suppliers is 10 hundred million dollars. This corresponds exactly to the area of the shaded region.

iii. Since the area calculated in question (a) is 13 hundred million dollars, the consumer surplus for espresso machines is $(13 - 10) = 3$ hundred million dollars.

Since the maximum amount of money that consumers would pay for 2 million espresso machines is 13 hundred million dollars, this means that consumers save 3 hundred million dollars when buying 2 million espresso machines at a fixed price set by the market equilibrium.

iv. Since the area calculated in question (b) is 3.6667 hundred million dollars, the supplier surplus for espresso machines is $(10 - 3.6667) = 6.3333$ hundred million dollars.

Since the minimum amount of money that suppliers would accept in exchange for 2 million espresso machines is 3.6667 hundred million dollars, this means that suppliers gain 6.3333 hundred million dollars when selling 2 million espresso machines at a fixed price set by the market equilibrium.

(d) Explain why a fixed price per unit (determined by the market equilibrium) is good for both consumers and suppliers (as opposed to the situations in questions (a) and (b)).

Solution: At the fixed price set by the market equilibrium, both consumers and suppliers stand to gain (since the consumer and supplier surpluses are both positive).

Moreover, the market equilibrium also ensures that the maximum possible number of espresso machines are sold, since to sell more than 2 million espresso machines (i.e. to the right of the market equilibrium), suppliers would require a price that is higher than what consumers are willing to pay.