Math 150 03 – Calculus I

In-class problem sheet

Thursday, October 4, 2023

1. Evaluate the derivatives of the following functions (over any interval where they are defined).

   (a) $f(x) = x^3 - 9x^2 + 16$ (b) $f(x) = \dfrac{x^3 + 25}{3x - 2}$ (c) $f(x) = \dfrac{x + 3}{x^2 - 4}$

Solution:

(a) We have $f(x) = g(x) - 9h(x) + 16$. Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= g'(x) - 9h'(x) + 0 && \text{(using the sum and product rules)} \\ &= 3x^2 - 9 \cdot 2x + 0 \\ &= \boxed{3x^2 - 18x} \end{aligned}$$

(b) We have $f(x) = \dfrac{g(x)}{h(x)}$. Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2} && \text{(using the quotient rule)} \\ &= \frac{3x^2 \cdot (3x - 2) - (x^3 + 25) \cdot 3}{(3x - 2)^2} \\ &= \frac{9x^3 - 6x^2 - 3x^3 - 75}{(3x - 2)^2} \\ &= \boxed{\frac{6x^3 - 6x^2 - 75}{(3x - 2)^2}} \end{aligned}$$

(c) We have $f(x) = \dfrac{g(x)}{h(x)}$. Using the rules for derivatives, we get:

$$\begin{aligned} f'(x) &= \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{(h(x))^2} && \text{(using the quotient rule)} \\ &= \frac{1 \cdot (x^2 - 4) - (x + 3) \cdot 2x}{(x^2 - 4)^2} \\ &= \frac{x^2 - 4 - 2x^2 - 6x}{(x^2 - 4)^2} \\ &= \boxed{\frac{-x^2 - 6x - 4}{(x^2 - 4)^2}} \end{aligned}$$

Remember: The **chain rule** says that if f and g are real functions, then the derivative of the composite function $(g \circ f)(x) = g(f(x))$ can be calculated as follows.

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

2. Evaluate the derivatives of the following functions.

(a) $f(x) = (x^2 + 3x + 4)^3$ (b) $f(x) = (x^3 + 5)^{1/4}$ (c) $f(x) = \dfrac{x+3}{(x^2-4)^{2/3}}$

Solution:

(a)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^3 \qquad \text{where } g(x) = x^2 + 3x + 4.$$

So if we define the function m as $m(x) = x^3$, then f is equal to the composite function $m \circ g$,

$$\text{i.e.} \qquad f(x) = (m \circ g)(x) = m(g(x)) = (g(x))^3 = \left(x^2 + 3x + 4\right)^3$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = (m \circ g)'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of the functions m and g.

$$\begin{aligned} m'(x) &= 3x^2 \\ g'(x) &= 2x + 3 \end{aligned}$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= 3(g(x))^2 \cdot (2x+3) && (\text{since } m'(x) = 3x^2 \text{ and } g'(x) = 2x+3) \\ &= \boxed{3(x^2 + 3x + 4)^2 \cdot (2x+3)} \end{aligned}$$

(b)

Step 1: We recognise that $f(x)$ can be rewritten as

$$f(x) = (g(x))^{1/4} = m(g(x)) = (m \circ g)(x) \qquad \text{where } g(x) = x^3 + 5 \text{ and } m(x) = x^{1/4}$$

Hence, using the chain rule, the derivative of f is

$$f'(x) = m'(g(x)) \cdot g'(x)$$

Step 2: We calculate the derivatives of m and g.

$$\begin{aligned} m'(x) &= \frac{1}{4} \cdot x^{-3/4} \qquad \text{(using the extended power rule)} \\ g'(x) &= 3x^2 \end{aligned}$$

Step 3: We calculate the derivative of f.

$$\begin{aligned} f'(x) &= m'(g(x)) \cdot g'(x) \\ &= \frac{1}{4} \cdot (g(x))^{-3/4} \cdot (3x^2) \qquad \text{(since } m'(x) = \frac{1}{4} \cdot x^{-3/4} \text{ and } g'(x) = 3x^2\text{)} \\ &= \boxed{\frac{1}{4} \cdot (x^3 + 5)^{-3/4} \cdot (3x^2)} \end{aligned}$$

(c) Using the quotient rule, we have

$$f'(x) = \frac{x \cdot (x^2 - 4)^{2/3} - (x + 3) \cdot g'(x)}{(x^2 - 4)^{4/3}}$$

where $g(x) = (x^2 - 4)^{2/3}$.

So we need to find the derivative of the function $g(x) = (x^2 - 4)^{2/3}$. First, we recognise that $g(x)$ can be rewritten as

$$g(x) = (k(x))^{2/3} = (m \circ k)(x) \qquad \text{where } k(x) = x^2 - 4 \text{ and } m(x) = x^{2/3}.$$

Hence, using the chain rule, the derivative of g is $g'(x) = m'(k(x)) \cdot k'(x)$.

Next, we calculate the derivatives of m and k.

$$\begin{aligned} m'(x) &= \frac{2}{3} \cdot x^{-1/3} \\ k'(x) &= 2x \end{aligned}$$

This lets us calculate the derivative of g.

$$\begin{aligned} g'(x) &= m'(k(x)) \cdot k'(x) \\ &= \frac{2}{3} \cdot (k(x))^{-1/3} \cdot (2x) \qquad \text{(since } m'(x) = \frac{2}{3} \cdot x^{-1/3} \text{ and k'(x)=2x)} \\ &= \frac{2}{3} \cdot \left(x^2 - 4\right)^{-1/3} \cdot (2x). \end{aligned}$$

Finally, we can calculate the derivative of f.

$$\begin{aligned} f'(x) &= \frac{x \cdot (x^2 - 4)^{2/3} - (x + 3) \cdot g'(x)}{(x^2 - 4)^{4/3}} \\ &= \boxed{\frac{x \cdot (x^2 - 4)^{2/3} - (x + 3) \cdot \frac{2}{3} \cdot \left(x^2 - 4\right)^{-1/3} \cdot (2x)}{(x^2 - 4)^{4/3}}} \end{aligned}$$

3. The distance between Los Angeles and San Diego on the I-5 highway is 118 miles.
   (a) What is the average speed (in mph) required to do the trip in 1.5 hours?
   (b) If the speed limit is 70 mph all along the I-5, is it possible to do the trip in 1.5 hours without breaking the law? Explain. What about in 2 hours?
   (c) A car going from L.A. to San Diego on the I-5 travels $f(x)$ miles (measured from L.A.) after x hours, where f is the function:
   $$f(x) = x(91 - 16x)$$
   i. How long does the car take to reach San Diego (i.e. cover 118 miles)?
   ii. Does the car ever break the law? (Is its speed ever more than 70 mph?)
   iii. What is the car's speed when it leaves L.A. (at the starting time)? What is the car's speed when it arrives in San Diego? (That is, at the time calculated in part i.)

Remember: Speed (or velocity) is the derivative of distance as a function of time.

Solution:

(a) The average speed required to do 118 miles in 1.5 hours is $\frac{118}{1.5} = \boxed{78.67}$ miles per hour (mph).

(b) Since the average speed required to do 118 miles in 1.5 hours is 78.67 mph, it is $\boxed{\text{not possible}}$ to stay below 70 mph and do the trip in 1.5 hours.

The average speed required to do 118 miles in 2 hours is $\frac{118}{2} = 59$ mph, so it is $\boxed{\text{possible}}$ to do the trip in 2 hours while staying below the speed limit.

(c) i. Since the car travels $f(x)$ miles in x hours, to find how many hours the car will take to do 118 miles, we need to solve the equation $f(x) = 118$,

$$\begin{aligned}
\text{i.e.} \quad & x(91 - 16x) = 118 \\
\text{i.e.} \quad & 91x - 16x^2 = 118 \\
\text{i.e.} \quad & 16x^2 - 91x + 118 = 0 \\
\text{i.e.} \quad & 16x^2 - 32x - 59x + 118 = 0 \\
\text{i.e.} \quad & 16x(x - 2) - 59(x - 2) = 0 \\
\text{i.e.} \quad & (16x - 59)(x - 2) = 0
\end{aligned}$$

The solutions of this equation are $x = 2$ and $x = \frac{59}{16} = 3.69$, so we can conclude that the car will be 118 miles from L.A. on the I-5 after 2 hours and after 3.69 hours, respectively. Since $2 < 3.69$, we can conclude that $\boxed{\text{the car will take 2 hours to do the trip.}}$

ii. The speed of the car at time x hours is the derivative $f'(x)$. We can use the rules for derivatives to calculate

$$\begin{aligned}
f'(x) &= 1 \cdot (91 - 16x) + x \cdot (0 - 16) \\
&= 91 - 16x - 16x \\
&= 91 - 32x.
\end{aligned}$$

So, at time 0 hours (the starting time in L.A.), the speed of the car is $f'(0) = 91-32\cdot 0 = \boxed{91}$ mph, and so the car **does** break the law.

iii. The speed of the car at the starting time ($x = 0$ hours) is $f'(0) = 91$ mph. Since the car takes 2 hours to do the trip to San Diego, the speed at the arrival time ($x = 2$ hours) is $f'(2) = 91 - 32 \cdot 2 = 91 - 64 = \boxed{27}$ mph.