Math 150 03 – Calculus I

Take-home test 2

Due: Wednesday, December 6, 2023, 11:59PM (**hard deadline**)

Instructions:

- This test has questions worth **20** points in total. In order to score 100%, you need to get **16** points in total.
- Any extra points (> 16) will eventually count towards increasing your grade ($A \to A^+$, $B^+ \to A$, $B^- \to B$, and so on) at the end of the semester.
- Each question is divided into subquestions. The points that each subquestion is worth are indicated next to it.
- Write your answers on separate sheets of paper.
- Write your name at the top of **each** page you use, and number each page.
- Number your answers correctly.
- **Justify each step in all your answers fully and clearly.** Answers with no explanation (*even if the calculation is correct*) are worth **zero** points. Answers with a full and correct explanation but a calculation error are worth more than 90% of the points.
- You are expected to work on this test **alone**. Plagiarism will be sanctioned with a fail grade.

1. (a) [2 points] Find the equation of the tangent line to the curve given by $x^3+y^2-2xy = 7$ at the point $(-1, 2)$.

   (b) [3 points] Find antiderivatives of the following functions, using the rules for antiderivatives.

      i. $h(x) = 3x\sqrt{1+x^2}$

      ii. $h(x) = \dfrac{\left(\sec\left(\frac{1}{x}\right)\right)^2}{x^2}$

2. (a) [3 points] Calculate the following definite integrals.

      i. $\displaystyle\int_0^1 \left(x^3 - x - e^x + 2\right) \cdot dx$

      ii. $\displaystyle\int_{-1}^0 \left(x^2 + \sqrt{x+1}\right) \cdot dx$

   (b) [2 points] Find two points where the curves $f(x) = x^4$ and $g(x) = 2 - x^2$ intersect and calculate the area of the region bounded by them.

3. Let $c(x)$ be the total cost (in thousands of dollars) for Chululi, Inc. to produce x thousand kilograms of hot sauce. Let $r(x)$ be the total revenue (in thousands of dollars) that Chululi receives from selling x thousand kilograms of hot sauce.

   (a) [1 point] The derivative $c'(x)$ is called the *marginal cost* function[1]. Suppose Chululi's marginal cost function is $c'(x) = 2x^{1/3}$. What does it cost Chululi (in thousands of dollars) to produce 9 thousand kilograms of hot sauce?

   (b) [1 point] The derivative $r'(x)$ is called the *marginal revenue* function[2]. Suppose Chululi's marginal revenue function is $r'(x) = 6x^{-1/6}$. What is Choluli's total revenue (in thousands of dollars) from selling 9 thousand kilograms of hot sauce?

   (c) [1 point] Calculate the area between the curves $f(x) = 6x^{-1/6}$ and $g(x) = 2x^{1/3}$ from $x = 0$ to $x = 9$.

   (d) [2 points] Chululi's profit from producing and selling x thousand kilograms of hot sauce is $p(x) = r(x) - c(x)$.

      i. How much hot sauce must Chululi produce and sell to maximize profit?

      ii. What is the maximum profit that Chululi can make from producing and selling hot sauce? Compare this to the answer in part (c).

[1] $c'(x)$ represents the cost (in dollars) to produce an additional kilogram of hot sauce after x thousand kilograms have already been produced.

[2] $r'(x)$ represents the revenue (in dollars) received from selling an additional kilogram of hot sauce after x thousand kilograms have already been sold.

Definition: A real function f is called a *probability distribution function* if it satisfies the following conditions:

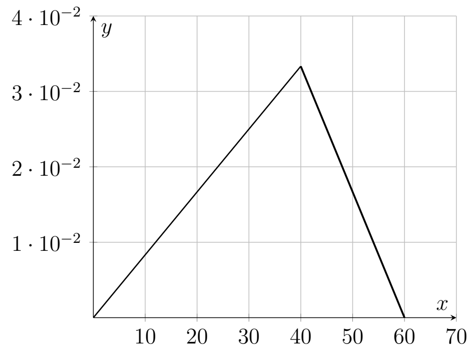
- For every real number x in the domain of f, we have $f(x) \geq 0$.
- The total area between the graph of f and the x axis is equal to 1.

Then the the definite integral $\int_a^b f(t) \cdot dt$ is the *probability* that x is in the interval $[a, b]$.

4. Consider the function defined as follows.

$$f(x) = \begin{cases} \dfrac{x}{1200} & \text{if } 0 \leq x \leq 40 \\ 0.1 - \dfrac{x}{600} & \text{if } 40 < x \leq 60 \end{cases}$$

(a) 3 points The graph of f is given below.



i. What is the domain of f?

ii. Explain why the definite integral $\int_0^{60} f(t) \cdot dt$ can be written as:

$$\int_0^{60} f(t) \cdot dt = \left(\int_0^{40} f(t) \cdot dt\right) + \left(\int_{40}^{60} f(t) \cdot dt\right)$$

iii. Show that f is a probability distribution function.

(b) 2 points Studies show that the previous function f is the probability distribution function associated to the time taken by a random student to finish a standardized 60-minute test. Therefore, the probability that a random student takes between 20 and 30 minutes to finish the test is given by the definite integral $\int_{20}^{30} f(t) \cdot dt = 0.2083$, or about 21%.

i. The *most probable outcome* (a.k.a. the *mode*) is the value $x = a$ such that $f(a)$ is the maximum value of f. Use the graph of f to find the most probable amount of time that a random student takes to finish the 60-minute test.

ii. The *expected value* (a.k.a the *mean*) is the definite integral $\int_a^b (t \cdot f(t)) \cdot dt$, where the interval $[a, b]$ is the domain of f. If a very large number (say a million) students take the 60-minute test, the expected value is the *average* amount of time that the students take to finish. Calculate the expected value of the distribution f, and compare it to the most probable outcome.