Math 150 03 – Calculus I

Homework assignment 2

Due: Wednesday, September 20, 2023

Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly. **Justify all your answers.**

1. Consider the following definition. (Recall that $|x|$ is the absolute value of x.)

$$f(x) = \frac{|x|}{x}$$

   (a) Does this define a real function? If so, what is its domain? Justify your answer.

   (b) Draw the graph of f.

   (c) Do the following limits exist? If so, what are they? Justify your answers.

      i. $\lim_{x \to -2^-} f(x)$ (the left-hand limit of f at -2.)

      ii. $\lim_{x \to 0^+} f(x)$ (the right-hand limit of f at 0.)

      iii. $\lim_{x \to 0} f(x)$

Solution:

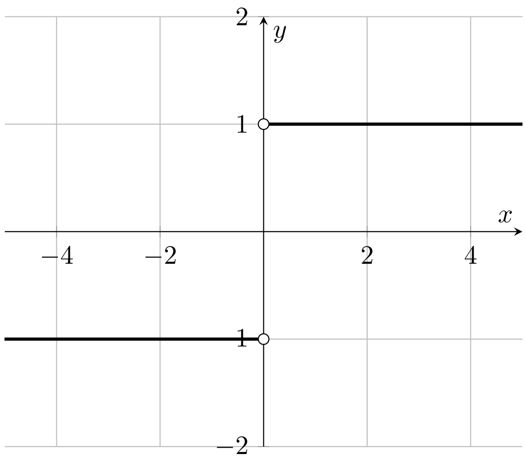
(a) For any real number x, *if* $f(x)$ is defined, then it has a unique value (passes the vertical line test). So f is a real function. However, $f(x)$ is not defined for all real numbers x.

We observe that

- When $x > 0$: $f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$. (Since when $x > 0$, $|x| = x$.)
- When $x = 0$: $f(0)$ is not defined.
- When $x < 0$: $f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$. (Since when $x < 0$, $|x| = -x$.)

Since $f(x)$ is defined for all real numbers except for $x = 0$, the domain of f is $(-\infty, 0) \cup (0, \infty)$.

We can graph f (for instance, in GeoGebra using the syntax "`f(x)=abs(x)/x`") to get a graph that looks like:



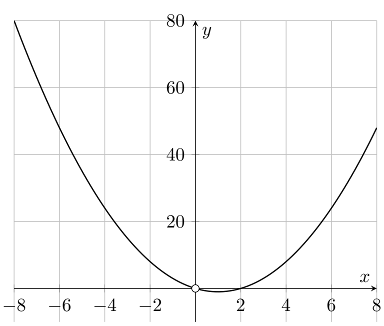
(b) i. From the graph, we observe that as x approaches -2 from the left, $f(x) = -1$ (i.e., $f(x)$ remains constant and equal to -1). So $\lim_{x\to 2^-} f(x) = -1$.

ii. From the graph, we observe that as x approaches 0 from the right, $f(x) = 1$ (i.e., $f(x)$ remains constant and equal to 1). So $\lim_{x\to 0^+} f(x) = 1$.

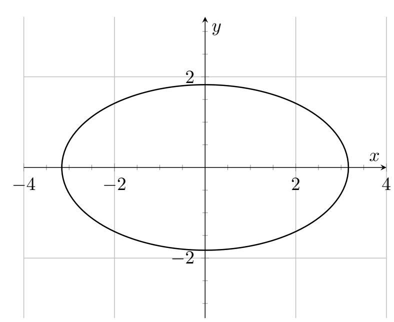
iii. We observe that as x approaches 0 from the left, $f(x) = -1$. This means that the left-hand limit of f at 0 is -1 (i.e., $\lim_{x\to 0^-} f(x) = -1$). Since the right-hand limit of f at 0 is 1 (i.e., $\lim_{x\to 0^+} f(x) = 1$) the left- and right-hand limits of f at 0 are not equal. Therefore the limit $\lim_{x\to 0} f(x)$ does not exist.

2. Which of the following are functions? What are their limits at $x = 0$? Are they continuous at $x = 0$?

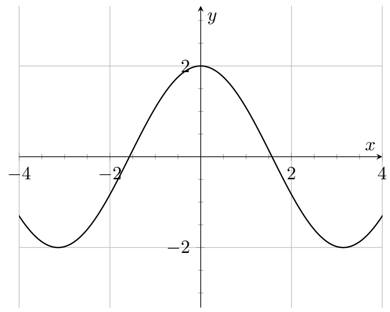
(a)



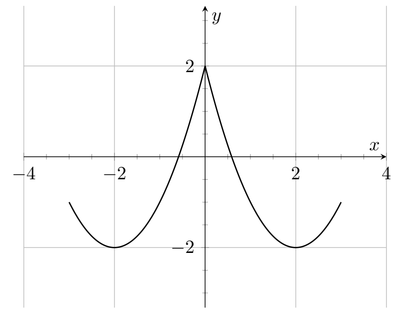
(b)



(c)

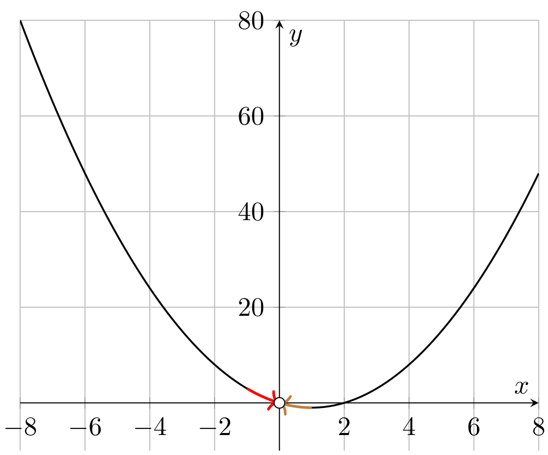


(d)



Solution:

(a)

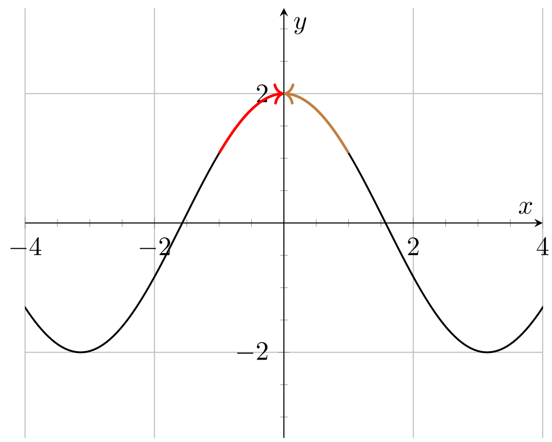


The graph is that of a real function, since it passes the vertical line test (but 0 is not in the domain of the function). We observe that as x approaches 0 from the left and the right, the graph approaches the same value (it approaches 0). In other words, the left-hand and right-hand limits at $x = 0$ exist and are both equal to 0. Therefore the limit of the function at $x = 0$ exists.

The function is not continuous at $x = 0$ because when $x = 0$, the value of the function is not defined. (However, since the function's limit at $x = 0$ *is* defined, the function has a *removable* discontinuity at $x = 0$).

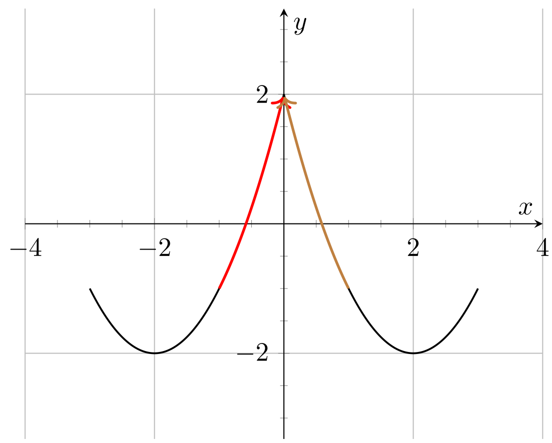
(b) The graph is not a real function, since it fails the vertical line test. So it doesn't make sense to talk about its limit at $x = 0$, or about it being continuous at $x = 0$.

(c)



The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value (it approaches 2). In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists. Since the function *is* defined at $x = 0$, and since its value at $x = 0$ is equal to its limit at $x = 0$, it is continuous at $x = 0$.

(d)



The graph is that of a real function, since it passes the vertical line test. We observe that as x approaches 0 from the left and the right, the graph approaches the same value (it approaches 2). In other words, the left-hand and right-hand limits at $x = 0$ exist and are equal. So the limit at $x = 0$ exists. Since the function *is* defined at $x = 0$, and since its value at $x = 0$ is equal to its limit at $x = 0$, it is continuous at $x = 0$.

3. Find the following limits. Justify your answers. (Hint: you can use a graphing calculator like **GeoGebra** to *see* what the functions look like, but you should give a complete justification of your answer.)

(a) $\lim_{x\to 0} x^2 + 4$

(b) $\lim_{x\to\infty} \frac{x^2}{x^2-1}$

(c) $\lim_{x\to 1} \frac{x^2+3x-4}{x^2-1}$

Solution:

(a) Since $x^2 + 4$ is a polynomial function, it always satisfies the substitution formula (using the sum and product rules for limits). So we have

$$\begin{aligned}\lim_{x\to 0} x^2 + 4 &= (0)^2 + 4 \\ &= 4\end{aligned}$$

(b) If we try to use the substitution formula, we get $\frac{\infty}{\infty}$ which is not a real number (doesn't make sense).

Whenever $x > 0$ (which will be the case when $x \to \infty$), we can divide by x^2 (since $x \neq 0$). So we have

$$\begin{aligned}\frac{x^2}{x^2-1} &= \frac{\frac{x^2}{x^2}}{\frac{x^2-1}{x^2}} \\ &= \frac{1}{1-\frac{1}{x^2}}.\end{aligned}$$

So we can use the algebra of limits to get

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} &= \frac{1}{1-\lim_{x\to\infty}\frac{1}{x^2}} \\ &= \frac{1}{1-0} = 1.\end{aligned}$$

(c) If we try to use the substitution formula we get $\frac{0}{0}$, which doesn't make sense. But this tells us that the polynomials $p(x) = x^2 + 3x - 4$ and $q(x) = x^2 - 1$ both have 1 as a root (that is, $p(1) = 0$ and $q(1) = 0$). Therefore $p(x) = (x-1) \cdot p_1(x)$ and $q(x) = (x-1) \cdot q_1(x)$. We can find $p_1(x)$ and $q_1(x)$ by long division:

$$\begin{array}{r|l} & x + 4 \\ \hline x - 1 & x^2 + 3x - 4 \\ & -x^2 + x \\ \hline & 4x - 4 \\ & -4x + 4 \\ \hline & 0 \end{array} \qquad \begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 \quad\; - 1 \\ & -x^2 + x \\ \hline & x - 1 \\ & -x + 1 \\ \hline & 0 \end{array}$$

So we have

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+3x-4}{x^2-1} &= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+4)}{\cancel{(x-1)}(x+1)} \\ &= \lim_{x\to 1}\frac{x+4}{x+1} = \frac{1+4}{1+1} = \frac{5}{2}.\end{aligned}$$

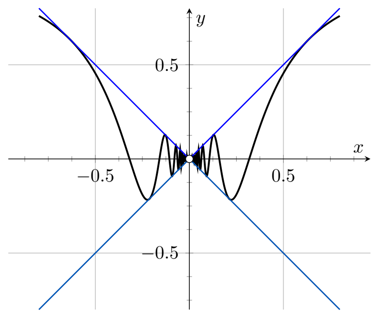
4. **(The squeeze theorem.)**

Theorem (Squeeze Theorem)**.** *Let $f, g, h \colon A \to \mathbb{R}$ be real functions such that for every $x \in A$, we have the inequalities $g(x) \le f(x) \le h(x)$. If $a, b \in \mathbb{R}$ are real numbers such that*

$$\lim_{x \to b} g(x) = \lim_{x \to b} h(x) = a,$$

then we have that $\lim_{x \to b} f(x) = a$.

We would like to use the squeeze theorem to show that $\lim_{x \to 0} \left(x \cdot \sin \frac{1}{x}\right) = 0$.



(a) What is the domain of the function $f(x) = x \cdot \sin \frac{1}{x}$?

(b) Show that, for any x in the domain of f, we have the inequalities $-|x| \le x \cdot \sin \frac{1}{x} \le |x|$.

(c) Use the graph of the absolute value function to show that $\lim_{x \to 0} |x| = 0$.

(d) Use the squeeze theorem to show that $\lim_{x \to 0} \left(x \cdot \sin \frac{1}{x}\right) = 0$.

Solution:

(a) The function f is defined at every real number x except $x = 0$ (since $\frac{1}{0}$ is not defined). Therefore, the domain of f is the set $(-\infty, 0) \cup (0, \infty)$.

(b) We need to show that when $x \in (-\infty, 0) \cup (0, \infty)$ then we have two inequalities: $-|x| \le x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \le |x|$.

Remember that we always have the inequalities $-1 \le \sin\left(\frac{1}{x}\right)$ and $\sin\left(\frac{1}{x}\right) \le 1$ (since the sine function always has values between -1 and 1).

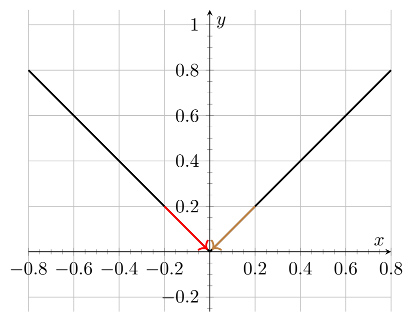
We can break up what we need to show into two cases.

1. When $x < 0$:
   - We can multiply both sides of the inequalities $-1 \le \sin\left(\frac{1}{x}\right)$ and $\sin\left(\frac{1}{x}\right) \le 1$ with x to get the inequalities $-x \ge x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \ge x$ (since multiplying both sides of an inequality with a negative number changes the direction of the inequality).
   - Next, when $x < 0$, we know that $|x| = -x$ and hence $-|x| = x$.
   - Therefore, we can rewrite the previous inequalities as $|x| \ge x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \ge -|x|$ respectively.
   - Since these are exactly the inequalities we're looking for, we are done in the case when $x < 0$.

2. When $x > 0$:
   - We can multiply both sides of the inequalities $-1 \leq \sin\left(\frac{1}{x}\right)$ and $\sin\left(\frac{1}{x}\right) \leq 1$ with x to get the inequalities $-x \leq x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \leq x$ (since multiplying both sides of an inequality with a positive number does not change the direction of the inequality).
   - Next, when $x > 0$, we know that $|x| = x$ and hence $-|x| = -x$.
   - Therefore, we can rewrite the previous inequalities as $-|x| \leq x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$ respectively.
   - Since these are exactly the inequalities we're looking for, we are done in the case when $x > 0$.

   Since we're done in both cases, we have shown the inequalities $-|x| \leq x \cdot \sin\left(\frac{1}{x}\right)$ and $x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$ for any $x \in (-\infty, 0) \cup (0, \infty)$.

(c) The graph of the absolute value function is given below.



We can see that as $x \to 0$ from the left and the right, the graph of $|x|$ goes to 0. Therefore the limit $\lim_{x \to 0} |x|$ is equal to 0.

(d) To apply the squeeze theorem, we need to find functions g and h such that for every x, we have the inequalities $g(x) \leq x \cdot \sin\left(\frac{1}{x}\right) \leq h(x)$, and such that $\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x)$.

- Since we have just shown that $-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$, we can choose g and h to be

$$g(x) = -|x| \qquad \text{and} \qquad h(x) = |x|$$

  so that we have $g(x) \leq x \cdot \sin\left(\frac{1}{x}\right) \leq h(x)$.
- Next, using the algebra of limits, we have

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} -|x| = -\lim_{x \to 0} |x| = -0 = 0$$

  and

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} |x| = 0.$$

Therefore, we can apply the squeeze theorem using the functions $g(x) = -|x|$ and $h(x) = |x|$ to get that $\lim_{x \to 0} f(x) = 0$.