Math 150 03 – Calculus I

Homework assignment 3

Due: Wednesday, October 4, 2023

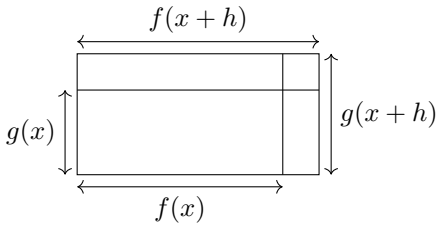
Instructions: Write your answers on a separate sheet of paper. Write your name at the top of each page you use, and number each page. Number your answers correctly.
Justify all your answers.

1. **(The product rule for derivatives.)**

   Let f and g be real functions that are differentiable at a real number x (i.e. such that the derivatives $f'(x)$ and $g'(x)$ exist). We would like to calculate the derivative $(f \cdot g)'(x)$ of the product function $f \cdot g$. Remember that this function is defined as:

   $$f \cdot g(x) = f(x) \cdot g(x)$$

   (a) Suppose that h is a small real number. In the figure below, shade the region that corresponds to the value $\big(f \cdot g(x+h) - f \cdot g(x)\big)$.

   (b) Use the figure to show that this value can also be written as

   $$\Big(\big(f(x+h) - f(x)\big) \cdot g(x+h)\Big) + \Big(f(x) \cdot \big(g(x+h) - g(x)\big)\Big)$$

   (c) Use the previous answer to show that the difference quotient $D(f \cdot g)_x(h)$ can be written as

   $$D(f \cdot g)_x(h) = \big(Df_x(h) \cdot g(x+h)\big) + \big(f(x) \cdot Dg_x(h)\big)$$

   (d) Since g is differentiable at x, g must be continuous at x (we will see why later). Show that if g is continuous at x, then we have that

   $$\lim_{h \to 0} g(x+h) = g(x)$$

   (e) Use the previous two answers and the algebra of limits to show the **product rule for derivatives**:

   $$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

2. (a) Let f be the function $f(x) = x^3$. Calculate the h-difference quotient $Df_x(h)$ and the derivative function f' using the limit definition of the derivative.

   (b) Since the function $f(x) = x^3$ can be written as $f(x) = x \cdot x^2$, use the product rule for derivatives to calculate the derivative $f'(x)$. Is this easier than the calculation in the previous question?

   (c) Use the product rule for derivatives to calculate the derivatives of the functions:

      i. $f(x) = x^4$ (Hint: $f(x) = x \cdot x^3$)

      ii. $f(x) = x^5$

      iii. $f(x) = x^6$

   (d) Infer a general formula for the derivative of the function $f(x) = x^n$, where $n \in \mathbb{N}$ is some natural number.

   (e) Use the previous answer to calculate the derivative of the function $f(x) = x^9 - 4x^3 + 7$.