Math 150 03 – Calculus I

Homework assignment 4

Due: Wednesday, October 18, 2023

The derivatives of some useful functions are given below.

- If $f(x) = a^x$ (for some constant real number $a \geq 0$), then $f'(x) = a^x \cdot \ln(a)$ (where $\ln(a) = \log_e a$).
- If $f(x) = \log_a x$ (for some constant real number $a > 0$ and $a \neq 1$), then $f'(x) = \dfrac{1}{x \cdot \ln(a)}$.
- If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.
- If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

1. Evaluate the derivatives of the following functions.

   (a) $f(x) = 2^{x^2+2x} \cdot (\cos(x))^4$ (b) $f(x) = (e^x + 3)^{\frac{1}{2}}$ (c) $f(x) = (\sec(x) + e^x)^9$

Solution:

(a) We have $f(x) = g(x) \cdot h(x)$, where $g(x) = 2^{x^2+2x}$ and $h(x) = (\cos(x))^4$. Hence

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x) \qquad \text{(using the product rule)}$$

To find $g'(x)$ we can use the chain rule by writing $g(x)$ as a composite function $g(x) = i(j(x))$, where $i(x) = 2^x$ and $j(x) = x^2 + 2x$. Hence

$$\begin{aligned} g'(x) &= i'(j(x)) \cdot j'(x) \\ &= \left(2^{x^2+2x} \cdot \ln(2)\right) \cdot (2x+2) \qquad \text{(since } i'(x) = 2^x \ln(2) \text{ and } j'(x) = 2x+2\text{)} \end{aligned}$$

Similarly, to find $h'(x)$, we can use the chain rule by writing $h(x)$ as a composite function $h(x) = i(j(x))$, where $i(x) = x^4$ and $j(x) = \cos(x)$. Hence

$$\begin{aligned} h'(x) &= i'(j(x)) \cdot j'(x) \\ &= 4(\cos(x))^3 \cdot (-\sin(x)) \qquad \text{(since } i'(x) = 4x^3 \text{ and } j'(x) = -\sin(x)\text{)} \\ &= -4(\cos(x))^3 \cdot \sin(x) \end{aligned}$$

Therefore, we can finally write $f'(x)$ as

$$\begin{aligned} f'(x) &= \left(\left(2^{x^2+2x} \cdot \ln(2)\right) \cdot (2x+2) \cdot (\cos(x))^4\right) + \left(2^{x^2+2x} \cdot (-4(\cos(x))^3 \cdot \sin(x))\right) \\ &= \boxed{2^{x^2+2x} \cdot (\cos(x))^3 \cdot \big(\ln(2) \cdot (2x+2) \cdot \cos(x) - 4\sin(x)\big)}. \end{aligned}$$

(b) We have $f(x) = g(h(x))$, where $g(x) = x^{\frac{1}{2}}$ and $h(x) = e^x + 3$. Hence we can use the chain rule to find

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= \boxed{\frac{1}{2} \cdot (e^x + 3)^{-\frac{1}{2}} \cdot e^x} \qquad \text{(since } g'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}} \text{ and } h'(x) = e^x\text{)} \end{aligned}$$

(c) We have $f(x) = g(h(x))$, where $g(x) = x^9$ and $h(x) = \sec(x) + e^x$. To find $h'(x)$, we have

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] + \frac{d}{dx}\left[e^x\right] \qquad \text{(using the sum rule, and since } \sec(x) = \frac{1}{\cos(x)}\text{)} \\ &= \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{(\cos(x))^2} + e^x \qquad \text{(using the quotient rule)} \\ &= \frac{\sin(x)}{(\cos(x))^2} + e^x \\ &= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} + e^x \\ &= \sec(x) \cdot \tan(x) + e^x \end{aligned}$$

Finally, we have

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \qquad \text{(using the chain rule)} \\ &= \boxed{9(\sec(x) + e^x)^8 \cdot (\sec(x) \cdot \tan(x) + e^x)} \qquad \text{(since } g'(x) = 9x^8\text{)} \end{aligned}$$

2. (Price elasticity of demand)

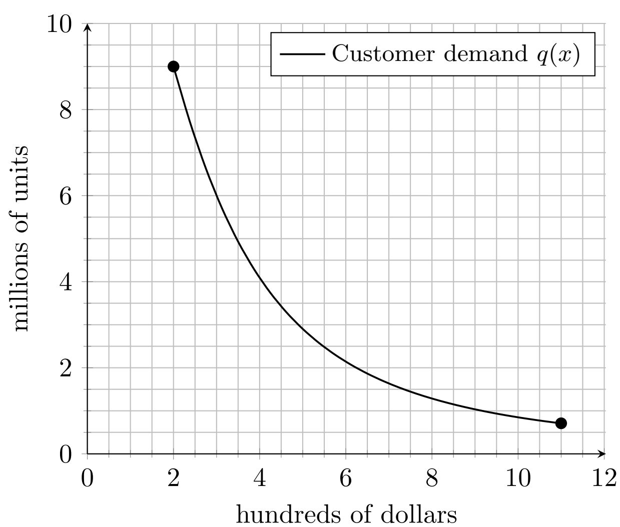

Figure 1: Market demand model

A tech company, Lemon Inc., plans to release a new cell phone, the piePhone 15. Prior to release, Lemon runs a market study that estimates the number of units of the piePhone 15 that they can expect to sell at a given price point (the “customer demand” graph). Lemon can see that customer demand ($q(x)$

million units sold at a price of x hundred dollars per unit) is a continuous function (over the interval $[2, 11]$), since it satisfies the vertical line test and the pen-to-paper test.

Obviously, Lemon's *revenue* from selling $q(x)$ million units at x hundred dollars each is $x \cdot q(x)$ hundred million dollars. That is, their revenue function is:

$$r(x) = x \cdot q(x) \quad \text{hundred million dollars.}$$

(a) Lemon wants to know the *marginal revenue** (i.e. the derivative r' of the revenue function r) in terms of the *marginal demand* (i.e. the derivative q' of the demand function q). Show that we can write the marginal revenue function as:

$$r'(x) = q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)$$

Remark: The function $E_d(x) = \frac{x}{q(x)} \cdot q'(x)$ is called the *price elasticity of demand*† (that some of you may have seen in an economics class). It is extremely important in economics — it measures how sensitive demand is to changes in price. $E_d(x)$ is almost always a negative real number (i.e. $E_d(x) < 0$). If $E_d(x) = -2$, it means that a 10% increase in price will result in a 20% *decrease* in demand.

(b) Lemon hires some pretty solid economists who figure out that the demand function q is given by:

$$q(x) = \frac{90}{x^2 + 6}$$

i. Calculate $q'(x)$, $E_d(x)$ and $r'(x)$.
ii. Calculate $r'(2)$. Is revenue increasing or decreasing at a price point of \$200 per unit?
iii. Calculate $r'(6)$. Is revenue increasing or decreasing at a price point of \$600 per unit?
iv. Find a price point a such that the revenue $r(a)$ is maximum.
v. What is $E_d(6)$? At a price point of \$600 per unit, how much (in %) will demand increase or decrease if the price increases by 7%?

Solution:

(a) Lemon's revenue function is $r(x) = x \cdot q(x)$. Using the rules for derivatives, the rate of change of revenue is

$$\begin{aligned} r'(x) &= \frac{d}{dx}[x \cdot q(x)] \\ &= \left(\frac{d}{dx}[x]\right) \cdot q(x) + x \cdot q'(x) && \text{(using the product rule)} \\ &= 1 \cdot q(x) + x \cdot q'(x) \\ &= \boxed{q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)} \end{aligned}$$

*Wikipedia article on marginal revenue.
†Wikipedia article on the price elasticity of demand.

(b) i. We can use the rules for derivatives to calculate

$$\begin{aligned} q'(x) &= \frac{d}{dx}\left[\frac{90}{x^2+6}\right] \\ &= \frac{\left(\frac{d}{dx}[90]\right)(x^2+6) - 90 \cdot \frac{d}{dx}\left[x^2+6\right]}{(x^2+6)^2} && \text{(using the quotient rule)} \\ &= \frac{0 \cdot (x^2+6) - 90 \cdot (2x+0)}{(x^2+6)^2} \\ &= \boxed{-\frac{180x}{(x^2+6)^2}} \end{aligned}$$

Now that we know $q(x)$ and $q'(x)$, we can calculate $r'(x)$ as:

$$\begin{aligned} r'(x) &= q(x) + x \cdot q'(x) \\ &= \frac{90}{x^2+6} + x \cdot \left(-\frac{180x}{(x^2+6)^2}\right) \\ &= \frac{90}{x^2+6} - \frac{180x^2}{(x^2+6)^2} \\ &= \frac{90(x^2+6) - 180x^2}{(x^2+6)^2} \\ &= \frac{90(x^2+6-2x^2)}{(x^2+6)^2} \\ &= \boxed{\frac{90(6-x^2)}{(x^2+6)^2}} \end{aligned}$$

ii. We have

$$r'(2) = \frac{90(6-(2)^2)}{((2)^2+6)^2} = \frac{90(6-4)}{(4+6)^2} = \frac{180}{100} = \boxed{1.8}$$

Since $r'(2) > 0$, Lemon's revenue (the function r) is $\boxed{\text{increasing}}$ at a price point of \$200 per unit.

iii. We have

$$r'(6) = \frac{90(6-(6)^2)}{((6)^2+6)^2} = \frac{90(-30)}{(42^2)} = \boxed{-\frac{75}{49}}$$

Since $r'(6) < 0$, Lemon's revenue is $\boxed{\text{decreasing}}$ at a price point of \$600 per unit.

iv. To find a price point a such that r has a maximum at a, we proceed as follows:

- First we solve $r'(x) = 0$ to find the critical points of r:

$$\begin{aligned} r'(x) &= 0 \\ \text{i.e.} \quad \frac{90(6-x^2)}{(x^2+6)^2} &= 0 \\ \text{i.e.} \quad 90(6-x^2) &= 0 \qquad \text{(multiplying both sides by } (x^2+6)^2\text{)} \\ \text{i.e.} \quad x^2 &= 6 \\ \text{i.e.} \quad x = \sqrt{6} \quad &\text{or} \quad x = -\sqrt{6}. \end{aligned}$$

Hence, if the revenue function r has a maximum, it must be at a price point of $\sqrt{6}$ hundred dollars or a price point of $-\sqrt{6}$ hundred dollars (unlikely, since in this situation, Lemon Inc. is *paying* customers to buy the piePhone 15).

- Next, we calculate $r''(x)$ and check whether $r'(x)$ is *decreasing* (i.e. $r''(x) < 0$) for each of the solutions from the previous step. We have

$$\begin{aligned}
r''(x) &= \frac{d}{dx}\left[\frac{90(6-x^2)}{(x^2+6)^2}\right] \\
&= \frac{\frac{d}{dx}\left[90(6-x^2)\right]\cdot(x^2+6)^2 - 90(6-x^2)\cdot\frac{d}{dx}\left[(x^2+6)^2\right]}{(x^2+6)^4} && \text{(using the quotient rule)} \\
&= \frac{(-180x)(x^2+6)^{\cancel{2}} - 90(6-x^2)\left(2\cancel{(x^2+6)}\cdot 2x\right)}{(x^2+6)^{\cancel{4}\ 3}} && \text{(using the product and chain rules)} \\
&= \frac{-180x((x^2+6)-2(6-x^2))}{(x^2+6)^3} \\
&= \frac{-180x(x^2+6-12+2x^2)}{(x^2+6)^3} \\
&= \frac{-180x(3x^2-6)}{(x^2+6)^3}
\end{aligned}$$

Hence we can calculate $r''(\sqrt{6})$ and $r''(-\sqrt{6})$ as follows.

$$\begin{aligned}
r''(\sqrt{6}) &= \frac{-180(\sqrt{6})(3(6)-6)}{(6+6)^3} = \frac{-180(\sqrt{6})(12)}{12^3} \\
\text{Hence}\qquad r''(\sqrt{6}) &< 0. \\
r''(-\sqrt{6}) &= \frac{-180(-\sqrt{6})(3(6)-6)}{(6+6)^3} = \frac{180(\sqrt{6})(12)}{12^3} \\
\text{Hence}\qquad r''(-\sqrt{6}) &> 0.
\end{aligned}$$

Since $r'(x)$ is *decreasing* when $x = \sqrt{6}$ and *increasing* when $x = -\sqrt{6}$, we know that r has a local maximum at $x = \sqrt{6}$ and a local minimum at $x = -\sqrt{6}$.

- Finally, since $x = \sqrt{6}$ is the only local maximum of r, we can conclude that the revenue is maximized at a price point of $\boxed{\sqrt{6}\text{ hundred dollars}}$ (approx. \$244.94).

v. We can calculate the price elasticity of demand at a price point of x hundred dollars per unit (i.e. the function $E_d(x)$) as follows.

$$\begin{aligned}
E_d(x) &= \frac{x}{q(x)}\cdot q'(x) \\
&= x\cdot\frac{(x^2+6)}{90}\cdot\left(-\frac{180x}{(x^2+6)^2}\right) \\
&= x\cdot\left(-\frac{2x}{x^2+6}\right) \\
&= -\frac{2x^2}{x^2+6}
\end{aligned}$$

Hence we have

$$E_d(6) = -\frac{2(6)^2}{(6)^2+6} = -\frac{2\cdot 6\cdot 6}{6\cdot 7} = \boxed{-\frac{12}{7}}$$

So, at a price point of \$600 per unit, if the price increases by 7%, the demand will change by

$$E_d(6) \cdot 7 = \left(-\frac{12}{7}\right) \cdot 7 = -12\%$$

In other words, the demand will $\boxed{\text{decrease by } 12\%}$.

3. (Variable interest rates.)

A lender (e.g. a bank) proposes two 20-year loans of \$1.5 million to a borrower (e.g. a homebuyer). The borrower can either pay *compound* interest at an interest rate of x percent, or *simple* interest at an interest rate of $2x$ percent. So, if the borrower chooses to pay compound interest at a rate of x percent, the amount they have to pay after 20 years is

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} \quad \text{million dollars.}$$

If the borrower chooses to pay simple interest, the amount they have to pay after 20 years is

$$g(x) = 1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right) \quad \text{million dollars.}$$

(a) If the compound interest rate is 4 percent, how much would the borrower pay after 20 years if they chose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(b) If the compound interest rate is 8 percent, how much would the borrower pay after 20 years if they choose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(c) The difference between the two loans is measured by the amount $d(x) = g(x) - f(x)$.

i. Calculate the rate of change of the difference between the two loans at a compound interest rate of 2 percent.

ii. Calculate the rate of change of the difference between the two loans at a compound interest rate of 4 percent.

iii. At what compound interest rate is the difference between the two loans *maximum*?

Solution:

(a) If the compound interest rate is 4 percent, we are in the case where $x = 4$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(4) = 1.5 \cdot \left(1 + \frac{4}{100}\right)^{20} = 3.287 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 4}{100}\right) = 3.9 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *less* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{compound interest}}$ makes more sense for the borrower.

(b) If the compound interest rate is 8 percent, we are in the case where $x = 8$.

i. If the borrower chooses the loan with compound interest, they would pay

$$f(8) = 1.5 \cdot \left(1 + \frac{8}{100}\right)^{20} = 6.991 \quad \text{million dollars.}$$

ii. If the borrower chooses the loan with simple interest, they would pay

$$g(4) = 1.5 \cdot \left(1 + 20 \cdot \frac{2 \cdot 8}{100}\right) = 6.3 \quad \text{million dollars.}$$

Since the amount paid on the loan with compound interest at 4 percent is *more* than the amount paid on the loan with simple interest at 8 percent, the loan with $\boxed{\text{simple interest}}$ makes more sense for the borrower.

(c) The rate of change of the difference function d is its derivative, which is $d'(x) = g'(x) - f'(x)$. We calculate the derivative of g as

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right)\right] \\ &= 1.5 \cdot \frac{d}{dx}\left[1 + 20 \cdot \frac{2x}{100}\right] \\ &= 1.5 \cdot \left(0 + 20 \cdot \frac{2}{100}\right) = \frac{6}{10}. \end{aligned}$$

To calculate the derivative of f, we recognise that $f(x)$ can be written as

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} = 1.5 \cdot (k(x))^{20} = m(k(x)) = (m \circ k)(x)$$

where $m(x) = 1.5 \cdot x^{20}$ and $k(x) = 1 + \frac{x}{100}$.

Hence we can find the derivative of f using the chain rule:

$$\begin{aligned} f'(x) &= m'(k(x)) \cdot k'(x) \\ &= 1.5 \cdot 20 \cdot (k(x))^{19} \cdot \frac{1}{100} && \text{since } m'(x) = 1.5 \cdot 20 \cdot x^{19} \text{ and } k'(x) = \frac{1}{100} \\ &= 30 \cdot \left(1 + \frac{x}{100}\right)^{19} \cdot \frac{1}{100} \\ &= \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \end{aligned}$$

Finally, we calculate $d'(x)$ as

$$\begin{aligned} d'(x) &= g'(x) - f'(x) \\ &= \frac{6}{10} - \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} \\ &= \frac{3}{10} \cdot \left(2 - \left(1 + \frac{x}{100}\right)^{19}\right) \end{aligned}$$

i. We have

$$d'(2) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{2}{100}\right)^{19}\right) = 0.163 \quad \text{million dollars per percent}$$

ii. We have

$$d'(4) = \frac{3}{10} \cdot \left(2 - \left(1 + \frac{4}{100}\right)^{19}\right) = -0.032 \quad \text{million dollars per percent}$$

iii. Any real number a is a local maximum of the difference function d exactly when $d'(a) = 0$ and $d''(a) < 0$.

Step 1: We find the critical points of the function d by solving the equation $d'(x) = 0$.

$$\begin{aligned}
& d'(x) = 0 \\
\text{i.e.} \quad & g'(x) - f'(x) = 0 \\
\text{i.e.} \quad & f'(x) = g'(x) \\
\text{i.e.} \quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = \frac{6}{10} \\
\text{i.e.} \quad & \frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19} = 2 \\
\text{i.e.} \quad & 1 + \frac{x}{100} = 2^{1/19} \\
\text{i.e.} \quad & x = 100 \cdot \left(2^{1/19} - 1\right) \\
\text{i.e.} \quad & x = 3.715
\end{aligned}$$

Step 2: The second derivative of d is the derivative of d', which we calculate as

$$\begin{aligned}
d''(x) &= g''(x) - f''(x) \\
&= \frac{d}{dx}\left[\frac{6}{10}\right] - \frac{d}{dx}\left[\frac{3}{10} \cdot \left(1 + \frac{x}{100}\right)^{19}\right] \\
&= 0 - \frac{3}{10}\frac{d}{dx}\left[\left(1 + \frac{x}{100}\right)^{19}\right]
\end{aligned}$$

Using the chain rule, $\frac{d}{dx}\left[\left(1 + \frac{x}{100}\right)^{19}\right] = 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$

So $d''(x) = -\frac{3}{10} \cdot 19 \cdot \left(1 + \frac{x}{100}\right)^{18} \cdot \frac{1}{100}$

Therefore, $d''(3.715) < 0$ (since $3.715 > 0$).

Hence we can conclude that $x = 3.715$ is the *only* local maximum of d. Therefore, the difference between the two loans is maximum when the compound interest rate is 3.715 percent.