Math 150 03 – Calculus I

Homework assignment 6

Due: Wednesday, November 8, 2023

1. Find antiderivatives of the following functions using the Anti-Sum and Anti-Constant Multiple rules.
   (a) $\dfrac{1}{3}x^5 + x^3 - 4$
   (b) $\dfrac{x^2 - 3x + 2}{x^2}$

Solution:

(a)

1. We have
$$f(x) = g(x) + h(x) + i(x)$$
where $g(x) = \dfrac{1}{3}x^5$, $h(x) = x^3$, and $i(x) = -4$. Using the Anti-Sum rule, we have that $\left(\int f\right)(x) = \left(\int g\right)(x) + \left(\int h\right)(x) + \left(\int i\right)(x)$.
2. To find $\left(\int g\right)(x)$, we use the anti-constant multiple rule and the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1} + C$ (whenever $a \neq -1$) to get
$$\left(\int g\right)(x) = \frac{1}{3} \cdot \frac{x^6}{6} + C = \frac{x^6}{18} + C$$
3. To find $\left(\int h\right)(x)$, we use the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1}+C$ (whenever $a \neq -1$) to get
$$\left(\int h\right)(x) = \frac{x^4}{4} + C$$
4. To find $\left(\int i\right)(x)$, we use the anti-constant multiple rule and the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1} + C$ (whenever $a \neq -1$) to get
$$\left(\int i\right)(x) = -4x + C$$
5. Therefore,
$$\boxed{\left(\int f\right)(x) = \frac{x^6}{18} + \frac{x^4}{4} - 4x + C}$$

(b)

1. We have
$$f(x) = \frac{x^2 - 3x + 2}{x^2} = 1 - \frac{3}{x} + \frac{2}{x^2} = g(x) + h(x) + i(x)$$
where $g(x) = 1$, $h(x) = -\frac{3}{x}$, and $i(x) = \frac{2}{x^2}$. Using the Anti-Sum rule, we have that $\left(\int f\right)(x) = \left(\int g\right)(x) + \left(\int h\right)(x) + \left(\int i\right)(x)$.

2. To find $\left(\int g\right)(x)$, we use the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1}+C$ (whenever $a \neq -1$) to get

$$\left(\int g\right)(x) = x + C$$

3. To find $\left(\int h\right)(x)$, we use the anti-constant multiple rule and the fact that an antiderivative of $\frac{1}{x}$ is $\ln(x) + C$ to get

$$\left(\int h\right)(x) = -3\ln(x) + C$$

4. To find $\left(\int i\right)(x)$, we use the anti-constant multiple rule and the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1} + C$ (whenever $a \neq -1$) to get

$$\left(\int i\right)(x) = 2 \cdot \frac{x^{-1}}{-1} + C = -\frac{2}{x} + C$$

5. Therefore,

$$\boxed{\left(\int f\right)(x) = x - 3\ln(x) - \frac{2}{x} + C}$$

Recall that the Anti-Chain rule can be stated as follows.

Anti-Chain Rule: "If $h(x) = g'(f(x)) \cdot f'(x)$, then the antiderivative of h is $\left(\int h\right)(x) = g(f(x)) + C$, where C is any constant real number."

2. (a) Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Use the Anti-Chain rule to calculate the antiderivative of $h(x) = \tan(x)$.
   (b) Calculate $\frac{d}{dx}[\tan(x)]$.
   (c) Recall that $\sec(x)^2 = \tan(x)^2 + 1$. Use this and the previous answer to calculate the antiderivative of $h(x) = \tan(x)^2$.

Solution:

(a) Let $h(x) = \tan(x)$. We want to find functions g and such that $g'(f(x)) \cdot f'(x) = h(x)$. Since we have that $h(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} \cdot \sin(x)$ we can let $f(x) = \cos(x)$ (since $\cos(x)$ is the "inside" function). Then $f'(x) = -\sin(x)$, and so we can solve $g'(\cos(x)) \cdot (-\sin(x)) = \frac{\sin(x)}{\cos(x)}$ for g'.

$$\begin{aligned} g'(\cos(x)) \cdot (-\sin(x)) &= \frac{\sin(x)}{\cos(x)} \\ \text{i.e.} \qquad g'(\cos(x)) &= -\frac{1}{\cos(x)} \qquad \text{(dividing both sides by } -\sin(x)\text{)} \\ \text{i.e.} \qquad g'(x) &= -\frac{1}{x} \end{aligned}$$

Since $g'(x) = -\frac{1}{x}$, we can assume that $g(x) = -\ln(x)$. Then the Anti-Chain rule says that

$$\left(\int h\right)(x) = g(f(x)) + C = \boxed{-\ln(\cos(x)) + C}$$

(b) To calculate $\frac{d}{dx}[\tan(x)]$ we can use the quotient rule.

$$\begin{aligned}\frac{d}{dx}[\tan(x)] &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\ &= \frac{\frac{d}{dx}[\sin(x)]\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}[\cos(x)]}{\cos(x)^2} \\ &= \frac{\cos(x)^2 - \sin(x)\cdot(-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\ &= \frac{1}{\cos(x)^2} = \boxed{\sec(x)^2} \qquad (\text{since } \sin(x)^2+\cos(x)^2 = 1)\end{aligned}$$

(c) Using the trigonometric identity $(\sec(x)^2 = \tan(x)^2 + 1)$ have that $h(x) = \tan(x)^2 = \sec(x)^2 - 1 = f(x) + g(x)$, where $f(x) = \sec(x)^2$ and $g(x) = -1$.

Therefore, we can apply the Anti-Sum rule to get $\left(\int h\right)(x) = \left(\int f\right)(x) + \left(\int g\right)(x)$.

In part (b) we have shown that $f(x) = \sec(x)^2$ is the derivative of $\tan(x)$. Therefore the *anti*derivative of f is $\left(\int f\right)(x) = \tan(x) + C$.

We can use the anti-constant multiple rule and the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1} + C$ (whenever $a \neq -1$) to get $\left(\int g\right)(x) = -x + C$.

Therefore we have

$$\left(\int h\right)(x) = \left(\int f\right)(x) + \left(\int g\right)(x) = \boxed{\tan(x) - x + C}$$

Recall that the Anti-Chain rule can also be stated as follows.

Anti-Chain Rule: "If $h(x) = g(f(x)) \cdot f'(x)$, then the antiderivative of h is $\left(\int h\right)(x) = \left(\int g\right)(f(x))$, where $\left(\int g\right)$ is the antiderivative of g."

3. Write each of the following functions as $g(f(x)) \cdot f'(x)$ for an appropriate choice of the functions g and f. Use the Anti-Chain rule to evaluate their antiderivatives.

   (a) $h(x) = \frac{(\ln(x))^8}{x}$

   (b) $h(x) = \frac{e^{2x}}{3 + e^{2x}}$

   (c) $h(x) = \frac{e^{(3\sqrt{x})}}{\sqrt{x}}$

Solution:

(a) 1. We want to find $g(x)$ and $f(x)$ such that $h(x) = \frac{(\ln(x))^8}{x} = g(f(x)) \cdot f'(x)$. We recognise that the "inside" function is $\ln(x)$, therefore we let $f(x) = \ln(x)$.

2. Since $f(x) = \ln(x)$, we have $f'(x) = \frac{1}{x}$. Therefore we can substitute $h(x)$, $f(x)$ and $f'(x)$ in $g(f(x)) \cdot f'(x) = h(x)$ and solve for $g(x)$.

$$
\begin{aligned}
g(\ln(x)) \cdot \frac{1}{x} &= \frac{(\ln(x))^8}{x} \\
\text{i.e.} \quad g(\ln(x)) &= (\ln(x))^8 \qquad \text{(multiplying both sides by } x\text{)} \\
\text{i.e.} \quad g(x) &= x^8
\end{aligned}
$$

Therefore $g(f(x)) \cdot f'(x) = (\ln(x))^8 \cdot \frac{1}{x} = h(x)$.

3. Hence to apply the Anti-Chain rule, which tells us that $\left(\int h\right)(x) = \left(\int g\right)(f(x))$, we need to find $\left(\int g\right)$. Since $g(x) = x^8$, we can use the fact that an antiderivative of x^a is $\frac{x^{a+1}}{a+1} + C$ (whenever $a \neq -1$) to get $\left(\int g\right)(x) = \frac{x^9}{9} + C$.

4. Therefore, applying the Anti-Chain rule, we get

$$\left(\int h\right)(x) = \left(\int g\right)(f(x)) = \boxed{\frac{\ln(x)^9}{9} + C}$$

(b) 1. We want to find $g(x)$ and $f(x)$ such that $h(x) = \frac{e^{2x}}{3+e^{2x}} = e^{2x}(3+e^{2x})^{-1} = g(f(x)) \cdot f'(x)$. We recognise that the "inside" function is $(3 + e^{2x})$, therefore we let $f(x) = 3 + e^{2x}$.

2. Since $f(x) = (3 + e^{2x})$, we have $f'(x) = 2e^{2x}$ (using the chain rule). Therefore we can substitute $h(x)$, $f(x)$ and $f'(x)$ in $g(f(x)) \cdot f'(x) = h(x)$ and solve for $g(x)$.

$$
\begin{aligned}
g(3 + e^{2x}) \cdot 2e^{2x} &= \frac{e^{2x}}{3 + e^{2x}} \\
\text{i.e.} \quad g(3 + e^{2x}) &= \frac{1}{2(3 + e^{2x})} \qquad \text{(dividing both sides by } 2e^{2x}\text{)} \\
\text{i.e.} \quad g(x) &= \frac{1}{2x}
\end{aligned}
$$

Therefore $g(f(x)) \cdot f'(x) = \dfrac{1}{2(3 + e^{2x})} \cdot 2e^{2x} = \dfrac{e^{2x}}{3 + e^{2x}} = h(x)$.

3. Hence to apply the Anti-Chain rule, which tells us that $\left(\int h\right)(x) = \left(\int g\right)(f(x))$, we need to find $\left(\int g\right)$. Since $g(x) = \frac{1}{2x} = \frac{1}{2} \cdot \frac{1}{x}$, we can use the anti-constant multiple rule and the fact that an antiderivative of $\frac{1}{x}$ is $\ln(x) + C$ to get $\left(\int g\right)(x) = \frac{1}{2}\ln(x) + C$.

4. Therefore, applying the Anti-Chain rule, we get

$$\left(\int h\right)(x) = \left(\int g\right)(f(x)) = \boxed{\frac{1}{2}\ln(3 + e^{2x}) + C}$$

(c) 1. We want to find $g(x)$ and $f(x)$ such that $h(x) = \dfrac{e^{(3\sqrt{x})}}{\sqrt{x}} = \dfrac{e^{(3x^{\frac{1}{2}})}}{x^{\frac{1}{2}}} = g(f(x)) \cdot f'(x)$. We recognise that the "inside" function is $(3x^{\frac{1}{2}})$, therefore we let $f(x) = 3x^{\frac{1}{2}}$.

2. Since $f(x) = 3x^{\frac{1}{2}}$, we have $f'(x) = 3 \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}$. Therefore we can substitute $h(x)$, $f(x)$ and $f'(x)$ in $g(f(x)) \cdot f'(x) = h(x)$ and solve for $g(x)$.

$$g(3x^{\frac{1}{2}}) \cdot \frac{3}{2}x^{-\frac{1}{2}} = \frac{e^{(3x^{\frac{1}{2}})}}{x^{\frac{1}{2}}}$$
$$\text{i.e.} \quad g(3x^{\frac{1}{2}}) \cdot \frac{3}{2} = e^{(3x^{\frac{1}{2}})} \quad \text{(multiplying both sides by } x^{\frac{1}{2}}\text{)}$$
$$\text{i.e.} \quad g(3x^{\frac{1}{2}}) = \frac{2}{3} \cdot e^{(3x^{\frac{1}{2}})} \quad \text{(multiplying both sides by } \frac{2}{3}\text{)}$$
$$\text{i.e.} \quad g(x) = \frac{2}{3} \cdot e^{x}$$

Therefore $g(f(x)) \cdot f'(x) = \frac{2}{3} \cdot e^{(3x^{\frac{1}{2}})} \cdot \frac{3}{2}x^{-\frac{1}{2}} = \frac{e^{(3x^{\frac{1}{2}})}}{x^{\frac{1}{2}}} = h(x)$.

3. Hence to apply the Anti-Chain rule, which tells us that $\left(\int h\right)(x) = \left(\int g\right)(f(x))$, we need to find $\left(\int g\right)$. Since $g(x) = \frac{2}{3} \cdot e^{x}$, we can use the anti-constant multiple rule and the fact that an antiderivative of e^x is $e^x + C$ to get $\left(\int g\right)(x) = \frac{2}{3} \cdot e^x + C$.

4. Therefore, applying the Anti-Chain rule, we get

$$\left(\int h\right)(x) = \left(\int g\right)(f(x)) = \boxed{\frac{2}{3} \cdot e^{(3x^{\frac{1}{2}})} + C}$$

4. Let $h(x) = x\sqrt{x-1}$. We would like to find the antiderivative of h.

   (a) Let $f(x) = x - 1$. Show that we can write h as $h(x) = \left((f(x)+1) \cdot f(x)^{1/2}\right) \cdot f'(x)$.

   (b) Expand the previous expression to show that we can write h as

$$h(x) = i(f(x)) \cdot f'(x) + j(f(x)) \cdot f'(x)$$

   (c) Use the Anti-Sum and Anti-Chain rules to calculate the antiderivative of h.

Solution:

(a) Since $f(x) = x - 1$, we have that $f'(x) = 1$. Substituting for $f(x)$ and $f'(x)$, we get

$$\left((f(x)+1) \cdot f(x)^{1/2}\right) \cdot f'(x) = ((x-1)+1) \cdot (x-1)^{1/2} \cdot 1 = x\sqrt{x-1} = h(x)$$

(b) We have

$$\begin{aligned} h(x) &= \left((f(x)+1) \cdot f(x)^{1/2}\right) \cdot f'(x) \\ &= \left(f(x) \cdot f(x)^{1/2} + f(x)^{1/2}\right) \cdot f'(x) \\ &= f(x)^{3/2} \cdot f'(x) + f(x)^{1/2} \cdot f'(x) \\ &= i(f(x)) \cdot f'(x) + j(f(x)) \cdot f'(x) \end{aligned}$$

where $i(x) = x^{3/2}$ and $j(x) = x^{1/2}$.

(c) Therefore, by the Anti-Chain and Anti-Sum rules, we have that

$$\begin{aligned}
\left(\textstyle\int h\right)(x) &= \left(\textstyle\int i\right)(f(x)) + \left(\textstyle\int j\right)(f(x)) \\
&= \left(\frac{f(x)^{5/2}}{\frac{5}{2}} + C_1\right) + \left(\frac{f(x)^{3/2}}{\frac{3}{2}} + C_2\right) \qquad \text{(since an antiderivative of } x^a \text{ is } \frac{x^{a+1}}{a+1}\text{)} \\
&= \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}
\end{aligned}$$