Math 150 03 – Calculus I

Homework assignment 4

Due: Wednesday, October 18, 2023

The derivatives of some useful functions are given below.

If $f(x) = a^x$ (for some constant real number $a \geq 0$), then $f'(x) = a^x \cdot \ln(a)$ (where $\ln(a) = \log_e a$).

If $f(x) = \log_a x$ (for some constant real number $a > 0$ and $a \neq 1$), then $f'(x) = \dfrac{1}{x \cdot \ln(a)}$.

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

1. Evaluate the derivatives of the following functions.

   (a) $f(x) = 2^{x^2+2x} \cdot (\cos(x))^4$ (b) $f(x) = (e^x + 3)^{\frac{1}{2}}$ (c) $f(x) = (\sec(x) + e^x)^9$

2. (Price elasticity of demand)

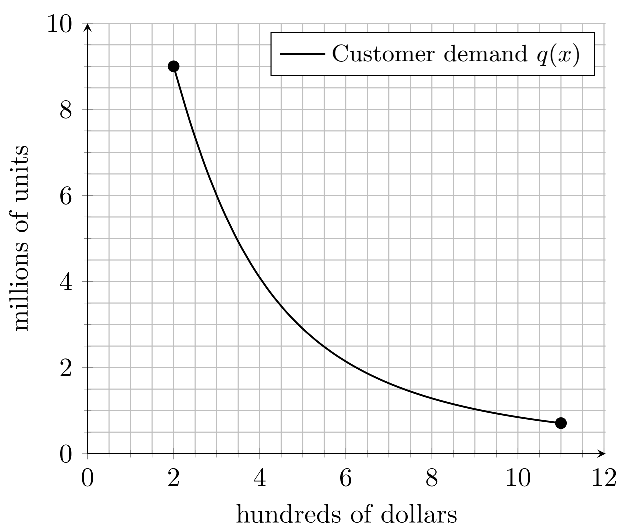

Figure 1: Market demand model

A tech company, Lemon Inc., plans to release a new cell phone, the piePhone 15. Prior to release, Lemon runs a market study that estimates the number of units of the piePhone 15 that they can expect to sell at a given price point (the "customer demand" graph). Lemon can see that customer demand ($q(x)$ million units sold at a price of x hundred dollars per unit) is a continuous function (over the interval $[2, 11]$), since it satisfies the vertical line test and the pen-to-paper test.

(a) $f(x) = 2^{x^2+2x} \cdot (\cos(x))^4$

Find $f'(x)$

$f(x) = g(x) \cdot h(x)$ where

$g(x) = 2^{x^2+2x}$ & $h(x) = (\cos(x))^4$

Using the prod rule for derivatives, we know that

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$$

so we need to find $g'(x)$ and $h'(x)$

To find $g'(x)$: We want to use the chain rule:

We write $g(x) = g_2 \circ g_1(x) = g_2(g_1(x))$

where $g_2(x) = 2^x$; $g_2'(x) = 2^x \cdot \ln(2)$

$g_1(x) = x^2 + 2x$; $g_1'(x) = 2x+2$

Applying the chain rule,

$$g'(x) = g_2'(g_1(x)) \cdot g_1'(x)$$

$$= 2^{g_1(x)} \cdot \ln(2) \cdot (2x+2)$$

so $g'(x) = 2^{x^2+2x} \cdot \ln(2) \cdot (2x+2)$

To find $h'(x)$: $h(x) = (\cos(x))^4$

We want to use the chain rule.

We see that

$$h(x) = i \circ j(x) = i(j(x))$$

Power rule: if $f(x) = x^a$ then $f'(x) = ax^{a-1}$

where $i(x) = x^4$; $i'(x) = 4x^3$

$j(x) = \cos(x)$; $j'(x) = -\sin(x)$

Applying the chain rule,

$$h'(x) = i'(j(x)) \cdot j'(x)$$
$$= 4(j(x))^3 \cdot (-\sin(x))$$

$$h'(x) = -4(\cos(x))^3 \cdot \sin(x)$$

Since $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

$$\boxed{\begin{aligned} f'(x) &= \left(2^{x^2+2x} \cdot \ln(2) \cdot (2x+2)\right) \cdot (\cos(x))^4 \\ &+ \left(2^{x^2+2x}\right) \cdot \left(-4(\cos(x))^3 \cdot \sin(x)\right) \end{aligned}}$$

(c) $f(x) = (\sec(x) + e^x)^9$

Find $f'(x)$

We want to use the chain rule.

We write $f(x) = (\sec(x) + e^x)^9 = g(\sec(x) + e^x)$

$$= g(h(x)) = g \circ h(x)$$

where $g(x) = x^9$; $g'(x) = 9x^8$

$$h(x) = \sec(x) + e^x$$

$$= \frac{1}{\cos(x)} + e^x$$

To find $h'(x)$, write $h(x) = i(x) + j(x)$

where $i(x) = \frac{1}{\cos(x)}$; $i'(x) = \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{(\cos(x))^2}$

$$= \frac{\sin(x)}{(\cos(x))^2}$$

$$= \sec(x) \cdot \tan(x)$$

$j(x) = e^x$; $j'(x) = e^x$

$$h'(x) = i'(x) + j'(x)$$

$$= \sec(x) \cdot \tan(x) + e^x$$

So $f'(x) = g'(h(x)) \cdot h'(x)$

$$= 9(h(x))^8 \cdot (\sec(x) \cdot \tan(x) + e^x)$$

$$= 9(\sec(x) + e^x)^8 \cdot (\sec(x) \cdot \tan(x) + e^x)$$

Obviously, Lemon's *revenue* from selling $q(x)$ million units at x hundred dollars each is $x \cdot q(x)$ hundred million dollars. That is, their revenue function is:

$$r(x) = x \cdot q(x) \quad \text{hundred million dollars.}$$

(a) Lemon wants to know the *marginal revenue** (i.e. the derivative r' of the revenue function r) in terms of the *marginal demand* (i.e. the derivative q' of the demand function q). Show that we can write the marginal revenue function as:

$$r'(x) = q(x) \cdot \left(1 + \frac{x}{q(x)} \cdot q'(x)\right)$$

Remark: The function $E_d(x) = \frac{x}{q(x)} \cdot q'(x)$ is called the *price elasticity of demand*† (that some of you may have seen in an economics class). It is extremely important in economics — it measures how sensitive demand is to changes in price. $E_d(x)$ is almost always a negative real number (i.e. $E_d(x) < 0$). If $E_d(x) = -2$, it means that a 10% increase in price will result in a 20% *decrease* in demand.

(b) Lemon hires some pretty solid economists who figure out that the demand function q is given by:

$$q(x) = \frac{90}{x^2 + 6}$$

i. Calculate $q'(x)$, $E_d(x)$ and $r'(x)$.

ii. Calculate $r'(2)$. Is revenue increasing or decreasing at a price point of \$200 per unit?

iii. Calculate $r'(6)$. Is revenue increasing or decreasing at a price point of \$600 per unit?

iv. Find a price point a such that the revenue $r(a)$ is maximum.

v. What is $E_d(6)$? At a price point of \$600 per unit, how much (in %) will demand increase or decrease if the price increases by 7%?

3. (Variable interest rates.)

A lender (e.g. a bank) proposes two 20-year loans of \$1.5 million to a borrower (e.g. a homebuyer). The borrower can either pay *compound* interest at an interest rate of x percent, or *simple* interest at an interest rate of $2x$ percent. So, if the borrower chooses to pay compound interest at a rate of x percent, the amount they have to pay after 20 years is

$$f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20} \quad \text{million dollars.}$$

If the borrower chooses to pay simple interest, the amount they have to pay after 20 years is

$$g(x) = 1.5 \cdot \left(1 + 20 \cdot \frac{2x}{100}\right) \quad \text{million dollars.}$$

(a) If the compound interest rate is 4 percent, how much would the borrower pay after 20 years if they chose the loan with:

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(b) If the compound interest rate is 8 percent, how much would the borrower pay after 20 years if they choose the loan with:

*Wikipedia article on marginal revenue.

†Wikipedia article on the price elasticity of demand.

i. Compound interest.

ii. Simple interest.

Which loan makes more sense for the homebuyer?

(c) The difference between the two loans is measured by the amount $d(x) = g(x) - f(x)$.

i. Calculate the rate of change of the difference between the two loans at a compound interest rate of 2 percent. i.e. find $d'(2)$

ii. Calculate the rate of change of the difference between the two loans at a compound interest rate of 4 percent. i.e. find $d'(4)$

iii. At what compound interest rate is the difference between the two loans *maximum*?

$$f(x) = 1.5\left(1 + \frac{x}{100}\right)^{20}$$

$$g(x) = 1.5\left(1 + \frac{20 \cdot 2x}{100}\right)$$

$$f(x) = 1.5\left(1 + \frac{x}{100}\right)^{20} \qquad d(x) = g(x) - f(x)$$

$$g(x) = 1.5\left(1 + \frac{20 \cdot 2x}{100}\right)$$

(i) Find $d'(2)$

First calculate $d'(x) = g'(x) - f'(x)$

To find $g'(x)$: $\quad g(x) = 1.5 + \frac{(1.5) \cdot 20 \cdot 2x}{100}$

so $\quad g'(x) = 0 + \frac{(1.5) \cdot 2\cancel{0} \cdot 2}{10\cancel{0}}$

$$= \frac{6}{10} = 0.6$$

To find $f'(x)$: $\quad f(x) = 1.5 \cdot \left(1 + \frac{x}{100}\right)^{20}$

$$= 1.5 \cdot i(j(x))$$

where $\quad i(x) = x^{20} \; ; \; 20x^{19}$

$\quad j(x) = 1 + \frac{x}{100} \; ; \; j'(x) = \frac{1}{100}$

Applying the prod. & chain rules,

$$f'(x) = 1.5\left(i'(j(x)) \cdot j'(x)\right)$$

$$= 1.5 \cdot 20 \cdot (j(x))^{19} \cdot \frac{1}{100}$$

$$= 1.5 \cdot 2\cancel{0} \cdot \left(1 + \frac{x}{100}\right)^{19} \cdot \frac{1}{10\cancel{0}} = \frac{3}{10}\left(1 + \frac{x}{100}\right)^{19}$$

so $d'(x) = g'(x) - f'(x)$

$$= 0.6 - \frac{3}{10}\left(1 + \frac{x}{100}\right)^{19}$$

so $d'(2) = 0.6 - \frac{3}{10}\left(1 + \frac{2}{100}\right)^{19}$

= use a calc.

Q (iii) Find a maximum of $d(x)$

Step 1: Solve $d'(x) = 0$ for x

$$d'(x) = 0.6 - \frac{3}{10}\left(1 + \frac{x}{100}\right)^{19}$$

so $d'(x) = 0$ is

$$\frac{6}{10} - \frac{3}{10}\left(1 + \frac{x}{100}\right)^{19} = 0$$

i.e. $$\frac{3}{\cancel{10}}\left(1 + \frac{x}{100}\right)^{19} = \frac{\cancel{6}^{\,2}}{\cancel{10}}$$

$$1 + \frac{x}{100} = \sqrt[19]{2}$$

$$x = \left(\sqrt[19]{2} - 1\right) \cdot 100$$

Step 2: Check that $d''\left(\left(\sqrt[19]{2} - 1\right) \cdot 100\right) < 0$