Math 150 03 – Calculus I

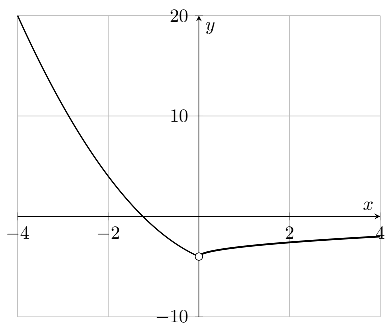
Take-home test 1

Due: Monday, October 9, 2023, 11:59PM (**hard deadline**)

Instructions:

- This test has questions worth **20** points in total. In order to score 100%, you need to get **16** points in total.
- Any extra points (> 16) will eventually count towards increasing your grade (A→A^+, B^+→A, B^-→B, and so on) at the end of the semester.
- Each question is divided into subquestions. The points that each subquestion is worth are indicated next to it.
- Write your answers on separate sheets of paper.
- Write your name at the top of **each** page you use, and number each page.
- Number your answers correctly.
- **Justify each step in all your answers fully and clearly.** Answers with no explanation (*even if the calculation is correct*) are worth **zero** points. Answers with a full and correct explanation but a calculation error are worth more than 90% of the points.
- You are expected to work on this test **alone**. Plagiarism will be sanctioned with a fail grade.

1. (a) $\boxed{\text{2 points}}$ Which of the following graphs represent real functions? Which of the functions is continuous over the interval $[-1, 1]$? Which of the functions has a removable discontinuity in the interval $[-1, 1]$?

i.


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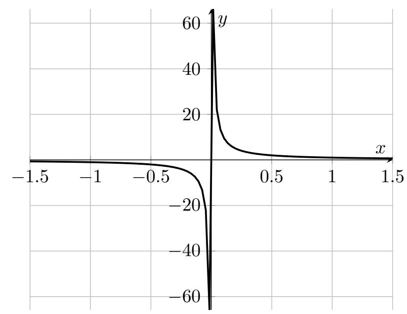
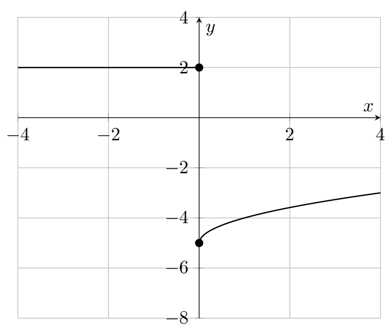


ii.


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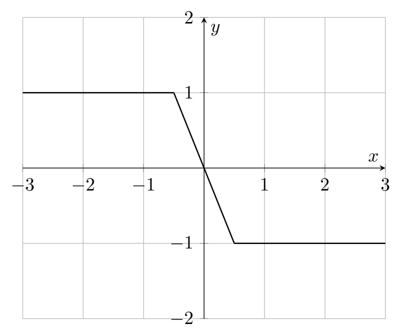


(b) $\boxed{\text{3 points}}$ Calculate the following limits. (Hint: you can use a graphing calculator to see what the functions look like, but you should also be able to justify your calculation.)

i. $\displaystyle\lim_{x\to -1} \frac{x^3 + 3x^2 + x - 1}{x + 1}$

ii. $\displaystyle\lim_{x\to \infty} \frac{13x^2}{x^3 + 2x^2 + 5x}$

2. Consider the following definition.

$$f(x) = \begin{cases} \dfrac{2}{x^5 + 2x^2 - 1} & \text{if } x \leq 0 \\ x^3 + 4x - 2 & \text{if } x > 0 \end{cases}$$

(a) $\boxed{\text{1.5 points}}$ Is f a real function? If so, what is its domain?

(b) $\boxed{\text{1.5 points}}$ Do the following limits exist? Justify your answer. If they do exist, calculate them.

i. $\displaystyle\lim_{x\to -1} f(x)$

ii. $\displaystyle\lim_{x\to 0^-} f(x)$

iii. $\displaystyle\lim_{x\to 0} f(x)$

(c) $\boxed{\text{2 points}}$ Is f continuous at 0?

(d) $\boxed{\text{2 points}}$ Consider the definition $g(x) = (7x^3 + 10x^2 + 5x + 2) \cdot f(x)$.

i. When $x \leq 0$, is $g(x)$ equal to a rational function?

ii. Calculate $\displaystyle\lim_{x\to -1} g(x)$

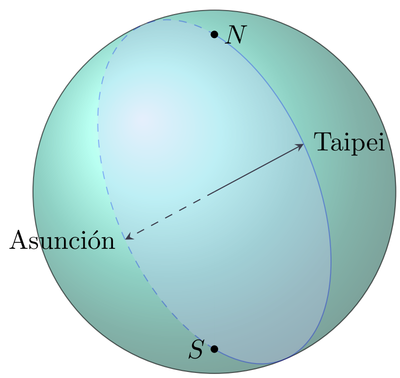


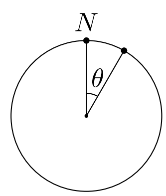
Figure 1: The Earth as a sphere, with the meridian passing through Taipei and Asunción drawn in blue.

3. If A is a point on the surface of a sphere, and if you draw a straight line from A passing through the center of the sphere, the point on the surface of the sphere at which you exit on the other side is called the **antipode** of A. For example, the Earth is (more or less) a sphere, and if you start at Taipei in Taiwan and drill a hole straight through the center of the Earth, you will exit at Asunción in Paraguay — so we say that Taipei and Asunción are a pair of antipodes. Similarly, the North and South poles are another pair of antipodes.

   A **great circle** is a circle drawn on the surface of a sphere such that for every point on the circle, its antipode is also on the circle. For example, on the Earth, the great circles passing through the North and South poles are called **meridians**.*

   (a) A (slightly inebriated) mathematician in a bar offers you \$100 if you can do the following: the mathematician gets to pick a meridian on the Earth, and you have to show that at any given instant in time, there is a pair of antipodes on that meridian that have the exact same temperature. If you fail (and if the mathematician can show you how to do it) you have to buy the mathematician a beverage of their choice.†

   Since the meridian is a circle passing through the North and South poles, every point on it is given by an angle θ from the North pole N.



   i. $\boxed{\text{1 point}}$ At a fixed instant in time, let $f(\theta)$ be the temperature at the point on the meridian that is at an angle θ from the North pole. So for example, $f(0°)$ is the temperature at the North pole, and $f(180°)$ is the temperature at the South pole. For any $\theta \in [0°, 180°]$, what is the temperature at the antipode of the point at θ?

   ii. $\boxed{\text{1 point}}$ Since the temperature at the Earth's surface doesn't have sudden jumps, we can assume that f is a continuous function. Let g be the following function.

$$g(\theta) = \text{ (temperature at } \theta \text{ on the meridian)} - \text{(temperature at } \theta\text{'s antipode)}$$

   Show that g is also a continuous function. (Hint: use the algebra of continuous functions.)

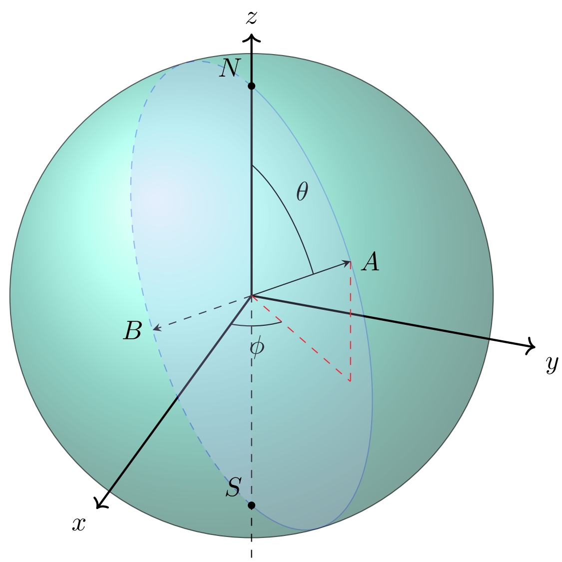
   iii. $\boxed{\text{2 points}}$ Use the intermediate value theorem to show that there is some $\theta \in [0°, 180°]$ such that $g(\theta) = 0$. Why does this say that there is a pair of antipodes on the meridian that have the exact same temperature?

*Each meridian is divided into two half-circles called *longitudes*.

†For the purposes of this question, imagine that you are of legal drinking age or vacationing in a more civilized country.

(b) Annoyed that you've won the bet, the mathematician offers you another challenge for "squares or nothing" (if you succeed, you get \$10,000 but if you fail, you lose nothing). You have to show that at any given instant in time, there is a pair of antipodes somewhere on the Earth that have the exact same temperature *and* the exact same humidity!

You know that mathematicians are unreliable (and they rarely have any money anyway), but you have nothing to lose, right?



Imagine that the Earth is a sphere centered at the origin in three-dimensional space. Then any point A on its surface can be specified by two angles (as in the figure above): the point $A = (\theta, \phi)$ must lie on some meridian, whose plane lies at an angle $\phi \in [0°, 360°]$ from the (x, z)-plane, and on this meridian, A lies at an angle $\theta \in [0°, 180°]$ from the North pole.[‡] On the Earth, the angles θ and ϕ are usually called the "latitude" and "longitude" respectively.

i. $\boxed{\text{0.5 points}}$ If the coordinates of the point A are $(50°, 50°)$, what are the coordinates of its antipode B?

ii. $\boxed{\text{1.5 points}}$ Any meridian on the Earth is given by a plane that is at an angle ϕ from the (x, z)-plane. In question 3(a) you showed that for every $\phi \in [0°, 360°]$, the meridian given by ϕ has a point at $(\theta(\phi), \phi)$ whose temperature is the same as its antipode's temperature. Consider the function $t\colon [0°, 360°] \to [0°, 180°]$ defined as follows.

$$t(\phi) = \begin{cases} \theta(\phi) & \text{if } 0° \le \phi < 180° \\ \\ 180° - \theta(\phi - 180°) & \text{if } 180° \le \phi < 360° \\ 180° - t(180°) & \text{if } \phi = 360° \end{cases}$$

Show that for every $\phi \in [0°, 180°]$, the points $(t(\phi), \phi)$ and $(t(180°+\phi), (180°+\phi))$ are antipodes and have the same temperature .

iii. $\boxed{\text{2 points}}$ For every $\phi \in [0°, 360°]$, let $h(\phi)$ be the humidity at the point $(t(\phi), \phi)$ on the Earth's surface. Since the temperature and humidity on the surface of the Earth vary continuously (there are no sudden jumps), we can assume that h is a continuous function. Show that $h(0°) = h(360°)$, and use the intermediate value theorem to show that there is some $\phi \in [0°, 180°]$ such that $h(\phi) = h(180° + \phi)$. Explain why this means that there is a point on the Earth's surface

[‡]These are also called "polar coordinates".

whose temperature and humidity are exactly the same as the temperature and humidity at its antipode.

Epilogue: You look up, after having spent hours solving the challenge, to find that the mathematician has slyly disappeared while you were working. All is not lost, however: at least you've managed to earn 4 bonus points on this take-home test, and you've proved the *Borsuk-Ulam theorem* in dimension 2!