Math 150 03 – Calculus I

Final Cheat-sheet

December 20, 2023

Algebra of limits

Let f and g be any real functions. If a is any real number or ∞ or $-\infty$, and if the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

- **Sum:** $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- **Product:** $\lim_{x\to a} f(x)\cdot g(x) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$
- **Quotient:** If $\lim_{x\to a} g(x)$ is not equal to 0, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Remember: A *rational function* is a real function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are 1-variable polynomials. The domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is the set of all real numbers *except* for the roots of q (the real numbers a such that $q(a) = 0$). If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, and if a is a root of *both* polynomials p and q, then $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{p_1(x)}{q_1(x)}$, where $p(x) = p_1(x)\cdot(x-a)$ and $q(x) = q_1(x)\cdot(x-a)$.

Algebra of continuity

Remember: A real function f is **continuous** at a real number a if $f(a)$ is defined and if $\lim_{x\to a} f(x) = f(a)$.

If f and g are continuous at a, then:

- **Sum:** $(f+g)(x) = f(x)+g(x)$ is continuous at a.
- **Product:** $(f\cdot g)(x) = f(x)\cdot g(x)$ is continuous at a.
- **Quotient:** If $g(a) \neq 0$, then $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$ is continuous at a.

Derivatives

Remember: Let f be a continuous real function. The *derivative* of f at a real number x is defined to be the limit:

$$f'(x) = \lim_{h\to 0} D_h f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

If the derivative of f exists at every real number x in an interval I, then f is **differentiable** over the interval I.

Remember: If f is differentiable over I, then the derivative f' is a real function that is *continuous* over I.

Remember: A function f is **strictly increasing** at a real number a if $f'(a) > 0$.

Remember: A function f is **strictly decreasing** at a real number a if $f'(a) < 0$.

Rules for derivatives

Let f, g be real functions that are differentiable over an interval I. Then,

- **Sum rule:**
$$(f+g)'(x) = f'(x) + g'(x)$$
- **Product rule:**
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
- **Quotient rule:** If $g(x)$ is never 0 over I, then
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

Some useful derivatives

The derivatives of some useful functions are given below.

- If $f(x) = a$ (for some constant real number a), then $f'(x) = 0$.
- If $f(x) = x^a$ (for some constant real number a), then $f'(x) = a \cdot x^{a-1}$.
- If $f(x) = a \cdot g(x)$ (for some constant real number a and some function g), then $f'(x) = a \cdot g'(x)$.
- If $f(x) = a^x$ (for some constant real number a, then $f'(x) = a^x \cdot \ln(a)$.
- If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.
- If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.
- If $f(x) = \tan(x)$, then $f'(x) = \sec(x)^2$.

Local and global maxima/minima

Let f be a differentiable real function.

- f has a *local maximum* at $x = a$ if $f'(a) = 0$ and if $f''(a) < 0$.
- f has a *local minimum* at $x = a$ if $f'(a) = 0$ and if $f''(a) > 0$.
- The *global maximum* of f is its largest local maximum.
- The *global minimum* of f is its smallest local maximum.

L'Hôpital's rule

When $x \to a$ (where a is any real number or $\pm\infty$), L'Hôpital's rule states that if $f(x)$ and $g(x)$ both approach 0 or both approach $\pm\infty$, then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
if the following conditions are satisfied:

1. the right hand limit exists,
2. if whenever $x \in (a-h, a) \cup (a, a+h)$ for some $h > 0$ (or, if $a = \pm\infty$, whenever $x \in (h, \infty)$ or $(-\infty, h)$ as the case may be), then $g'(x) \neq 0$ (that is, $g'(x) \neq 0$ when x is close to, but not equal to a).

Antiderivatives

An antiderivative of a real function f is a real function $\left(\int f\right)$ such that $\left(\int f\right)'(x) = f(x)$.

Some useful antiderivatives

- If $f(x) = x^a$, where a is any constant real number such that $a \neq -1$, then $\left(\int f\right)(x) = \dfrac{x^{a+1}}{a+1} + C$
- If $f(x) = \dfrac{1}{x}$, then $\left(\int f\right)(x) = \ln(x) + C$
- If $f(x) = e^{kx}$, where k is any constant real number such that $k \neq 0$, then $\left(\int f\right)(x) = \dfrac{1}{k}e^{kx} + C$
- If $f(x) = a^{kx}$, where a is any constant real number such that $a > 0$ and $a \neq 1$, and k is any constant real number such that $k \neq 0$, then $\left(\int f\right)(x) = \dfrac{1}{k\ln(a)}a^{kx} + C$
- If $f(x) = \ln(x)$, then $\left(\int f\right)(x) = x\ln(x) - x + C$
- If $f(x) = \sin(x)$, then $\left(\int f\right)(x) = -\cos(x) + C$
- If $f(x) = \cos(x)$, then $\left(\int f\right)(x) = \sin(x) + C$
- If $f(x) = \tan(x)$, then $\left(\int f\right)(x) = \ln(\sec(x)) + C$

Remark: in each of the previous expressions, C is any arbitrary constant.

Rules for antiderivatives

- **Anti-Sum rule:** If $h(x) = f(x) + g(x)$, then
$$\left(\int h\right)(x) = \left(\int f\right)(x) + \left(\int g\right)(x)$$
- **Anti-Constant Multiple rule:** If $g(x) = c \cdot f(x)$, where c is any constant real number, then
$$\left(\int g\right)(x) = c \cdot \left(\int f\right)(x)$$
- **Anti-Chain rule:** If $h(x) = g(f(x)) \cdot f'(x)$, then
$$\left(\int h\right)(x) = \left(\int g\right)(f(x))$$
where $\left(\int g\right)$ is an antiderivative of g.
- **Anti-Product rule:** If $h(x) = f(x) \cdot g'(x)$, then
$$\left(\int h\right)(x) = f(x) \cdot g(x) - \left(\int (f' \cdot g)\right)(x)$$
where $\left(\int (f' \cdot g)\right)$ is an antiderivative of the function $(f' \cdot g)(x) = f'(x) \cdot g(x)$.

Definite integrals

If f is a continuous real function, then the *definite integral of f over the interval* $[a, b]$ is the total area between the graph of f and the x-axis bounded by the points $x = a$ and $x = b$. The definite integral is written as $\displaystyle\int_a^b f(t) \cdot dt$.

Fundamental theorems of calculus

The *fundamental theorems of calculus* are the following facts.

FTC1 If f is a continuous real function, then
$$\boxed{F(x) = \int_0^x f(t) \cdot dt}$$
is an antiderivative of f.

FTC2 If f is a continuous real function and if $\left(\int f\right)$ is any antiderivative of f, then
$$\boxed{\int_a^b f(t) \cdot dt = \left(\int f\right)(b) - \left(\int f\right)(a)}$$